

## GROWTH IN FINITE SIMPLE GROUPS OF LIE TYPE

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ABSTRACT. We prove that if  $L$  is a finite simple group of Lie type and  $A$  a set of generators of  $L$ , then  $A$  grows i.e.  $|A^3| > |A|^{1+\varepsilon}$  where  $\varepsilon$  depends only on the Lie rank of  $L$ , or  $A^3 = L$ . This implies that for a family of simple groups  $L$  of Lie type the diameter of any Cayley graph is polylogarithmic in  $|L|$ . We also obtain some new families of expanders.

We also prove the following partial extension. Let  $G$  be a subgroup of  $GL(n, p)$ ,  $p$  a prime, and  $S$  a symmetric set of generators of  $G$  satisfying  $|S^3| \leq K|S|$  for some  $K$ . Then  $G$  has two normal subgroups  $H \geq P$  such that  $H/P$  is soluble,  $P$  is contained in  $S^6$  and  $S$  is covered by  $K^c$  cosets of  $H$  where  $c$  depends on  $n$ . We obtain results of similar flavour for sets generating infinite subgroups of  $GL(n, \mathbb{F})$ ,  $\mathbb{F}$  an arbitrary field.

## 1. INTRODUCTION

The diameter,  $\text{diam}(X)$ , of an undirected graph  $X = (V, E)$  is the largest distance between two of its vertices.

Given a subset  $A$  of the vertex set  $V$  the expansion of  $A$ ,  $c(A)$ , is defined to be the ratio  $|\sigma(A)|/|A|$  where  $\sigma(A)$  is the set of vertices at distance 1 from  $A$ . A graph is a  $C$ -expander for some  $C > 0$  if for all sets  $A$  with  $|A| < |V|/2$  we have  $c(A) \geq C$ . A family of graphs is an expander family if all of its members are  $C$ -expanders for some fixed positive constant  $C$ .

Let  $G$  be a finite group and  $S$  a symmetric (i.e. inverse-closed) set of generators of  $G$ . The Cayley graph  $\Gamma(G, S)$  is the graph whose vertices are the elements of  $G$  and which has an edge from  $x$  to  $y$  if and only if  $x = sy$  for some  $s \in S$ . Then the diameter of  $\Gamma$  is the smallest number  $d$  such that  $S^d = G$ .

The following classical conjecture is due to Babai [5]

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**Conjecture 1** (Babai). *For every non-abelian finite simple group  $L$  and every symmetric generating set  $S$  of  $L$  we have  $\text{diam}(\Gamma(L, S)) \leq C(\log |L|)^c$  where  $c$  and  $C$  are absolute constants.*

In a spectacular breakthrough Helfgott [29] proved that the conjecture holds for the family of groups  $L = PSL(2, p)$ ,  $p$  a prime. In recent major work [30] he proved the conjecture for the groups  $L = PSL(3, p)$ ,  $p$  a prime. Dinai [18] and Varjú [59] have extended Helfgott's original result to the groups  $PSL(2, q)$ ,  $q$  a prime power.

We prove the following.

**Theorem 2.** *Let  $L$  be a finite simple group of Lie type of rank  $r$ . For every symmetric set  $S$  of generators of  $L$  we have*

$$\text{diam}(\Gamma(L, S)) < (\log |L|)^{c(r)}$$

where the constant  $c(r)$  depends only on  $r$ .

This settles Babai's conjecture for any family of simple groups of Lie type of bounded rank.

A key result of Helfgott [29] shows that generating sets of  $SL(2, p)$  grow rapidly under multiplication. His bound on diameters is an immediate consequence.

**Theorem 3** (Helfgott). *Let  $L = SL(2, p)$  and  $A$  a generating set of  $L$ . Let  $\delta$  be a constant,  $0 < \delta < 1$ .*

a) *Assume that  $|A| < |L|^{1-\delta}$ . Then*

$$|A^3| \gg |A|^{1+\varepsilon}$$

where  $\varepsilon$  and the implied constant depend only on  $\delta$

b) *Assume that  $|A| > |L|^{1-\delta}$ . Then  $A^k = L$  where  $k$  depends only on  $\delta$ .*

It was observed in [50] that a result of Gowers [26] implies that b) holds for an arbitrary simple group of Lie type  $L$  with  $k = 3$  for some  $\delta(r)$  which depends only on the Lie rank  $r$  of  $L$  (see [4] for a more detailed discussion). Hence to complete the proof of our theorem on diameters it remains to prove an analogue of the (rather more difficult) part a) as was done by Helfgott for the groups  $SL(3, p)$  in [30].

We prove the following.

**Theorem 4.** *Let  $L$  be a finite simple group of Lie type of rank  $r$  and  $A$  a generating set of  $L$ . Then either  $A^3 = L$  or*

$$|A^3| \gg |A|^{1+\varepsilon}$$

where  $\varepsilon$  and the implied constant depend only on  $r$ .

We also give some examples which show that in the above result the dependence of  $\varepsilon$  on  $r$  is necessary. In particular we construct generating sets  $A$  of  $SL(n, 3)$  of size  $2^{n-1} + 4$  with  $|A^3| < 100|A|$  for  $n \geq 3$ .

Theorem 4 was first announced in [53]. The same day similar results were announced by Breuillard, Green and Tao [11] for finite Chevalley groups. It is noted in [11] that their methods are likely to extend to all simple groups of Lie type, but this has not yet been checked. On the other hand in [11] various interesting results for complex matrix groups were also announced.

Somewhat earlier Gill and Helfgott [24] had shown that small generating sets (of size at most  $p^{n+1-\delta}$  for some  $\delta > 0$ ) in  $SL(n, p)$  grow.

Helfgott's work [29] has been the starting point and inspiration of much recent work by Bourgain, Gamburd, Sarnak and others. Let  $S = \{g_1, g_2, \dots, g_k\}$  be a symmetric subset of  $SL(n, \mathbb{Z})$  and  $\Lambda = \langle S \rangle$  the subgroup generated by  $S$ . Assume that  $\Lambda$  is Zariski dense in  $SL(n)$ . According to the theorem of Matthews-Vaserstein-Weisfeiler [48] there is some integer  $m_0$  such that  $\pi_m(\Lambda) = SL(n, \mathbb{Z}/m\mathbb{Z})$  assuming  $(m, m_0) = 1$ . Here  $\pi_m$  denotes reduction mod  $m$ .

It was conjectured in [47], [9] that the Cayley graphs  $\Gamma(SL(n, \mathbb{Z}/m\mathbb{Z}), \pi_m(S))$  form an expander family, with expansion constant bounded below by a constant  $c = c(S)$ . This was verified in [6], [7], [9] in many cases when  $n = 2$  and in [8] for  $n > 2$  and moduli of the form  $p^d$  where  $d \rightarrow \infty$  and  $p$  is a sufficiently large prime.

In [8] Bourgain and Gamburd also prove the following

**Theorem 5** (Bourgain, Gamburd). *Assume that the analogue of Helfgott's theorem on growth holds for  $SL(n, p)$ ,  $p$  a prime. Let  $S$  be a symmetric finite subset of  $SL(n, \mathbb{Z})$  generating a subgroup  $\Lambda$  which is Zariski dense in  $SL(n)$ . Then the family of Cayley graphs  $\Gamma(SL(n, p), \pi_p(S))$  forms an expander family as  $p \rightarrow \infty$ . The expansion coefficients are bounded below by a positive number  $c(S) > 0$ .*

By Theorem 4 the condition of this theorem is satisfied hence the above conjecture is proved for prime moduli.

For  $n = 2$  Bourgain, Gamburd and Sarnak [9] proved that the conjecture holds for square free moduli. This result was used in [9] as a building block in a combinatorial sieve method for primes and almost primes on orbits of various subgroups of  $GL(2, \mathbb{Z})$  as they act on  $\mathbb{Z}^m$  (for  $m \geq 2$ ).

Recently, extending Theorem 5 P. Varjú [59] has shown that if the analogue of Helfgott's theorem holds for  $SL(n, p)$ ,  $p$  a prime, then the above conjecture holds for square free moduli and Zariski dense subgroups of  $SL(n)$ . Hence our results constitute a major step towards

obtaining a generalisation to Zariski dense subgroups of  $SL(n, \mathbb{Z})$  and to other arithmetic groups.<sup>1</sup>

Simple groups of Lie type can be treated as subgroups of simple algebraic groups. In fact, instead of concentrating on simple groups, we work in the framework of arbitrary linear algebraic groups over algebraically closed fields. We set up a machinery which can be used to obtain various results on growth of subsets in linear groups. In particular, we prove the following extension of Theorem 4, valid for finite groups obtained from connected linear groups over  $\overline{\mathbb{F}_p}$ , which produces growth within certain normal subgroups (for the terminology see Definition 66).

**Theorem 6.** *Let  $G$  be a connected linear algebraic group over  $\overline{\mathbb{F}_p}$  and  $\sigma : G \rightarrow G$  a Frobenius map. Let  $G^\sigma$  denote the subgroup of the fixpoints of  $\sigma$  and  $1 \in S \subseteq G^\sigma$  a symmetric generating set. Then for all  $1 > \varepsilon > 0$  there is an integer  $M = M_{\text{main}}(\dim(G), \varepsilon)$  and a real  $K$  depending on  $\varepsilon$  and the numerical invariants of  $G$  (notably  $\dim(G)$ ,  $\deg(G)$ ,  $\text{mult}(G)$  and  $\text{inv}(G)$ , see Definition 28) with the following property. If  $\mathcal{Z}(G)$  is finite and*

$$K \leq |S| \leq |G^\sigma|^{1-\varepsilon}$$

*then there is a connected closed normal subgroup  $H \triangleleft G$  such that  $\deg H \leq K$ ,  $\dim(H) > 0$  and*

$$|S^M \cap H| \geq |S|^{(1+\delta) \dim(H) / \dim(G)}$$

*where  $\delta = \frac{\varepsilon}{128 \dim(G)^3}$ .*

Consider the groups  $G^\sigma$  for simply connected simple algebraic groups  $G$ . Central extensions of all but finitely many simple groups of Lie type are obtained in this way (see [57]) and the centres  $\mathcal{Z}(G^\sigma)$  have bounded order. Hence Theorem 6 implies Theorem 4 for both twisted and untwisted simple groups of Lie type in a unified way.

The proof of Theorem 6 relies basically on two properties of the finite groups  $G^\sigma$ . First, if  $G^\sigma$  is large enough then  $\mathcal{C}_G(G^\sigma) = \mathcal{Z}(G)$ . Second, if a  $\sigma$ -invariant connected closed subgroup of  $G$  is normalised by  $G^\sigma$  then it is in fact normal in  $G$ . In this generality Theorem 6 depends on Hrushovski's twisted Lang-Weil estimates [31]. In the proof of Theorem 4 this can be avoided (see Remark 71). Hence the constants in this theorem are explicitly computable.

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<sup>1</sup>Finally the conjecture has very recently been proved by Bourgain and Varjú [13].

We believe that Theorem 6 and the general results concerning algebraic groups involved in its proof will have many applications to investigating growth in linear groups. Here we first prove (using Theorem 6) the following partial extension of Theorem 4:

**Theorem 7.** *Let  $S$  be a symmetric subset of  $GL(n, p)$  satisfying  $|S^3| \leq K|S|$  for some  $K \geq 1$ . Then  $GL(n, p)$  has two subgroups  $H \geq P$ , both normalised by  $S$ , such that  $P$  is perfect,  $H/P$  is soluble,  $P$  is contained in  $S^6$  and  $S$  is covered by  $K^{c(n)}$  cosets of  $H$  where  $c(n)$  depends on  $n$ .*

Understanding the structure of symmetric subsets  $S$  of  $GL(n, p)$  (or more generally of  $GL(n, q)$ ,  $q$  a prime-power) satisfying  $|S|^3 \leq K|S|$  is mentioned by Breuillard, Green and Tao as a difficult open problem in [11].

Subgroups of  $GL(n, p)$  generated by elements of order  $p$  were investigated in detail by Nori [49] and Hrushovski-Pillay [33]. As a byproduct of the proof of Theorem 7 we obtain the following.

**Theorem 8.** *Let  $P \leq GL(n, p)$ ,  $p$  a prime, be a perfect subgroup which is generated by its elements of order  $p$ . Let  $S$  be a symmetric set of generators of  $P$ . Then*

$$\text{diam}(\Gamma(P, S)) \leq (\log |P|)^{M(n)}$$

where the constant  $M(n)$  depends only on  $n$ .

Theorem 8 is a surprising extension of the fact (included in Theorem 2) that simple subgroups of  $GL(n, p)$  ( $n$  bounded) have polylogarithmic diameter.

Combining Theorem 8 with results of Aldous [1] and Babai [2] we immediately obtain the following corollary.

**Corollary 9.** *Let  $\Gamma = \Gamma(P, S)$  be a Cayley graph as in Theorem 8. Then  $\Gamma$  is a  $C$ -expander with some*

$$C \geq \frac{1}{1 + (\log |P|)^{M(n)}}.$$

Equivalently, if  $A$  is a subset of  $P$  of size at most  $|P|/2$ , then we have

$$|A \cdot S| \geq (1 + C)|A|.$$

For a very recent unexpected application in arithmetic geometry of the above corollary see [40].

To indicate the generality of our methods we derive the following consequence.

**Theorem 10.** *Let  $\mathbb{F}$  be an arbitrary field and  $S \subseteq GL(n, \mathbb{F})$  a finite symmetric subset such that  $|S^3| \leq K|S|$  for some  $K \geq \frac{3}{2}$ . Then there are normal subgroups  $H \leq \Gamma$  of  $\langle S \rangle$  and a bound  $m$  depending only on  $n$  such that  $\Gamma \subseteq S^6 H$ , the subset  $S$  can be covered by  $K^m$  cosets of  $\Gamma$ ,  $H$  is soluble, and the quotient group  $\Gamma/H$  is the product of finite simple groups of Lie type of the same characteristic as  $\mathbb{F}$ . (In particular, in characteristic 0 we have  $\Gamma = H$ .) Moreover, the Lie rank of the simple factors appearing in  $\Gamma/H$  is bounded by  $n$ , and the number of factors is also at most  $n$ .*

This theorem may be viewed as a common generalisation of Theorem 4 above and a result of Hrushovski [32] obtained by model-theoretic tools. It would be most interesting to obtain a result that would also imply Theorem 7.

The first result of this type was obtained by Elekes and Király [20]. In characteristic 0 the above theorem was first proved by Breuillard, Green and Tao [12]. Actually in that case they have a stronger conclusion: one can even require  $\Gamma = H$  to be nilpotent.

In earlier versions of our paper, for subsets of linear groups over infinite fields we only proved general results on growth. While writing the final version of this paper, we realised that Theorem 10 is a relatively easy consequence of these results.

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**1.1. Methods.** The proofs of Helfgott combine group theoretic arguments with some algebraic geometry, Lie theory and tools from additive combinatorics such as the sum-product theorem of Bourgain, Katz, Tao [10]. Our argument relies on a deeper understanding of the algebraic group theory behind his proofs and an extra trick, but not on additive combinatorics.

We prove various results which say that if  $L$  is a “nice” subgroup of an algebraic group  $G$  generated by a set  $A$  then  $A$  grows in some sense. These were motivated by earlier results of Helfgott [29], [30] and Hrushovski-Pillay [33].

To illustrate our strategy we outline the proof of Theorem 4 in the simplest case, when  $A$  generates  $L = SL(n, q)$ ,  $q$  a prime-power. Assume that “ $A$  does not grow” i.e.  $|AAA|$  is not much larger than  $|A|$ . Using an “escape from subvarieties” argument it is shown in [30] that if  $T$  is a maximal torus in  $L$  then  $|T \cap A|$  is not much larger than

$|A|^{1/(n+1)}$ . This is natural to expect for dimensional reasons since  $\dim(T)/\dim(L) = (n-1)/(n^2-1) = 1/(n+1)$ .

We use a rather more powerful escape argument. The first part of our paper is devoted to establishing the necessary tools in great generality (in particular Theorem 49).

Now  $T$  is equal to  $L \cap \bar{T}$  where  $\bar{T}$  is a maximal torus of the algebraic group  $SL(n, \overline{\mathbb{F}}_q)$ . Let  $T_r$  denote the set of regular semisimple elements in  $T$ . Note that  $T \setminus T_r$  is contained in a subvariety  $V \subsetneq \bar{T}$  of dimension  $n-2$ . By the above mentioned escape argument  $|(T \setminus T_r) \cap A|$  is not much larger than

$$|A|^{\dim(V)/\dim(L)} = |A|^{1/(n+1)-1/(n^2-1)}.$$

By [30] or by our escape argument  $A$  does contain regular semisimple elements. If  $a$  is such an element then consider the map  $SL(n) \rightarrow SL(n), g \rightarrow g^{-1}ag$ . The image of this map is contained in a subvariety of dimension  $n^2-1-(n-1)$  since  $\dim(\mathcal{C}_{SL(n)}(a)) = n-1$ . By the escape argument we obtain that for the conjugacy class  $\text{cl}(a)$  of  $a$  in  $L$ ,  $|\text{cl}(a) \cap A^{-1}aA|$  is not much larger than  $|A|^{(n^2-n)/(n^2-1)}$ . Now  $|\text{cl}(a) \cap A^{-1}aA|$  is at least the number of cosets of the centraliser  $C_L(a)$  which contain elements of  $A$ . It follows that  $|AA^{-1} \cap C_L(a)|$  is not much smaller than  $|A|^{1/(n+1)}$ . Of course  $C_L(a)$  is just the (unique) maximal torus containing  $a$ .

Let us say that  $A$  *covers* a maximal torus  $T$  if  $|T \cap A|$  contains a regular semisimple element. We obtain the following fundamental dichotomy (see Lemma 60):

*Assume that a generating set  $A$  does not grow*

- i) If  $A$  does not cover a maximal torus  $T$  then  $|T \cap A|$  is not much larger than  $|A|^{1/(n+1)-1/(n^2-1)}$ .*
- ii) If  $A$  covers  $T$  then  $|T \cap AA^{-1}|$  is not much smaller than  $|A|^{1/(n+1)}$ . In this latter case in fact  $|T_r \cap AA^{-1}|$  is not much smaller than  $|A|^{1/(n+1)}$ .*

It is well known that if  $A$  doesn't grow then  $B = AA^{-1}$  doesn't grow either hence the above dichotomy applies to  $B$ .

Let us first assume that  $B$  covers a maximal torus  $T$  but does not cover a conjugate  $T' = g^{-1}Tg$  of  $T$  for some element  $g$  of  $L$ . Since  $A$  generates  $L$  we have such a pair of conjugate tori where  $g$  is in fact an element of  $A$ . Consider those cosets of  $T'$  which intersect  $A$ . Each of the, say,  $t$  cosets contains at most  $|B \cap T'|$  elements of  $A$  i.e. not much more than  $|B|^{1/(n+1)-1/(n^2-1)}$  which in turn is not much

more than  $|A|^{1/(n+1)-1/(n^2-1)}$ . Therefore  $|A|$  is not much larger than  $t|A|^{1/(n+1)-1/(n^2-1)}$ .

On the other hand  $A(A^{-1}(BB^{-1})A)$  has at least  $t|T \cap BB^{-1}|$  elements which is not much smaller than  $t|A|^{1/(n+1)}$ . Therefore  $A(A^{-1}(AA^{-2}A)A)$  is not much smaller than  $|A|^{1+1/(n^2-1)}$  which contradicts the assumption that  $A$  does not grow.

We obtain that  $B$  covers all conjugates of some maximal torus  $T$ . Now the conjugates of the set  $T_r$  are pairwise disjoint (e.g. since two regular semisimple elements commute exactly if they are in the same maximal torus). The number of these tori is  $|L : N_L(T)| > c(n)|L : T|$  for some constant which depends only on  $n$ . Each of them contains not much less than  $|B|^{1/(n+1)}$  regular semisimple elements of  $BB^{-1}$ . Altogether we see that  $|A|$  is not much smaller than  $q^{n^2-n}|A|^{1/(n+1)}$  and finally that  $|A|$  is not much less than  $|L|$ . In this case by [50] we have  $AAA = L$ .

The proof of Theorem 6 follows a similar strategy. However there is an essential difference; maximal tori have to be replaced by a more general class of subgroups called CCC-subgroups (see Definition 57). These subgroups were in fact designed to make the argument work in not necessarily simple (or semisimple) algebraic groups. In Sections 8, 9 and 10 we establish the basic properties of these subgroups and justify that they indeed play the role of maximal tori in general algebraic groups. The proof of Theorem 6 is completed in Section 13.

In [49] Nori showed that if  $p$  is sufficiently large in terms of  $n$ , there is a correspondence between subgroups of  $GL(n, p)$  generated by elements of order  $p$  and a certain class of closed subgroups of  $GL(n, \overline{\mathbb{F}}_p)$ . Note that the bounds in [49] are ineffective. Using this correspondence Theorem 7 is proved for perfect  $p$ -generated groups by a short induction argument based on a slight extension of Theorem 6. The general case can be reduced to this by applying various known results on finite linear groups.

Theorem 10 follows by combining some of the ingredients of the proof of Theorem 7 in a rather more direct way.

Examples given in Section 14 show that in Theorem 4 we must have  $\varepsilon(r) = O(1/r)$ . We believe that this is the right order of magnitude.

## 2. NOTATION

Throughout this paper  $\overline{\mathbb{F}}$  denotes an arbitrary algebraically closed field. For a prime number  $p$  we denote by  $\mathbb{F}_p$  and  $\overline{\mathbb{F}}_p$  the finite field with  $p$  elements and its algebraic closure. Similarly,  $\mathbb{F}_q$  denotes the finite field with  $q$  elements, where  $q$  is a prime power. The letters  $N$



and  $\Delta$  will always be used for an upper bound for dimensions and degrees respectively,  $K$  is used for a lower bound on the size of certain finite sets. When we study growth,  $M$  will denote the length of the products we allow. In several lemmas we use a parameter  $\varepsilon$ , it is the error-margin we allow in the exponents when we count elements in certain subsets.

### 3. DIMENSION AND DEGREE

We use affine algebraic geometry i.e. all occurring sets will be subsets of some affine space  $\overline{\mathbb{F}}^m$  for some integer  $m > 0$ , and we define all of them via  $m$ -variate polynomials whose coefficients belong to  $\overline{\mathbb{F}}$ . Below we make this more precise.

**Definition 11.** A subset  $Z \subseteq \overline{\mathbb{F}}^m$  is *Zariski closed*, or simply *closed*, if it can be defined as the common zero set of some  $m$ -variate polynomials. This defines a topology on  $\overline{\mathbb{F}}^m$ , each subset of  $\overline{\mathbb{F}}^m$  inherits this topology, called the *Zariski topology*. This is the only topology that we use in this paper, so we omit the adjective Zariski. The complements of closed subsets are called *open*. The intersection of a closed and an open subset is called *locally closed*. If we do not use explicitly the ambient affine space then locally closed subsets are called *algebraic sets* and closed subsets are called *affine algebraic sets*. (Note, that our definition of algebraic set is rather restrictive.) For an arbitrary subset  $X \subseteq \overline{\mathbb{F}}^m$  we denote by  $\overline{X}$  the *closure* of  $X$ .

Note, that algebraic sets are always equipped (by definition) with an ambient affine space, even if it is not explicitly given. This is one reason for choosing the name “algebraic set” instead of “variety”.

**Definition 12.** An algebraic set  $X$  is called *irreducible* if it has the following property. Whenever  $X$  is contained in the union of finitely many closed subsets, it must be contained in one of them.

**Definition 13.** Let  $X$  be an algebraic set. Then there are finitely many closed subsets  $X_i \subseteq X$  which are irreducible, and maximal among the irreducible closed subsets of  $X$ . Then  $X = \bigcup_i X_i$  is the *irreducible decomposition* of  $X$  and these  $X_i$  are called the *irreducible components* of  $X$ .

**Definition 14.** Let  $Z \subseteq \overline{\mathbb{F}}^m$  be an algebraic set. We consider chains  $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$  where the  $Z_i$  are nonempty, irreducible closed subsets of  $Z$ . The largest possible length  $n$  of such a chain is called the *dimension* of  $Z$ , denoted by  $\dim(Z)$ .

**Definition 15.** Let  $X \subseteq \overline{\mathbb{F}}^m$  be an algebraic set. An *affine* subspace of  $\overline{\mathbb{F}}^m$  is a translate of a linear subspace. If  $X$  is irreducible then we consider all affine subspaces  $L \subseteq \overline{\mathbb{F}}^m$  such that  $\dim(X) + \dim(L) = m$  and  $X \cap L$  is finite. The *degree* of  $X$  is the largest possible number of intersection points:

$$\deg(X) = \max_L |X \cap L| .$$

In general, the degree of  $X$  is defined as the sum of the degrees of its irreducible components.

*Remark 16.* Let  $X$  be an algebraic set. Then  $\dim(X) = 0$  iff  $X$  is finite. A finite subset  $X \subset \overline{\mathbb{F}}^m$  is always closed, and satisfies  $\deg(X) = |X|$ .

**Definition 17.** Let  $X \subseteq \overline{\mathbb{F}}^m$  and  $Y \subseteq \overline{\mathbb{F}}^n$  be algebraic sets. A function  $f : X \rightarrow Y$  is called a *morphism* if it is the restriction to  $X$  of a map  $\phi : \overline{\mathbb{F}}^m \rightarrow \overline{\mathbb{F}}^n$  whose  $n$  coordinates are  $m$ -variate polynomials. Then the graph of  $f$ , denoted by  $\Gamma_f \subseteq X \times Y \subseteq \overline{\mathbb{F}}^{m+n}$ , is locally closed. We define the *degree* of  $f$  to be  $\deg(f) = \deg(\Gamma_f)$ .

*Remark 18.* Algebraic sets form a category with the above notion of morphism. Isomorphic algebraic sets have equal dimensions and isomorphisms respect the irreducible decomposition. In contrast, the degrees of isomorphic algebraic sets may not be equal.

In the present paper we work mainly in the category of algebraic sets and morphisms. To obtain explicit bounds we need to estimate the degrees of all appearing objects. If one is satisfied with existence results only then one can avoid all these calculations by simply noticing that all of our constructions can be done simultaneously in families of algebraic sets. (Such proofs a priori do not give explicit constants, but with careful examination, in principle they can be made explicit.) In fact this technique is really used e.g. in the proof of Proposition 80.

The following fact is standard:

**Fact 19.** Let  $X, Y \subseteq \overline{\mathbb{F}}^m$  be locally closed sets.

- (a) The dimension and the degree of  $X$  are equal to the dimension and the degree of its closure  $\overline{X}$ .
- (b) Any closed subset of  $X$  has dimension at most  $\dim(X)$ .
- (c) The irreducible components  $X_i \leq X$  satisfy

$$\dim(X_i) \leq \dim(X) = \max_j (\dim(X_j)) ,$$

$$\deg(X_i) \leq \deg(X) = \sum_j \deg(X_j) .$$

It follows that there are at most  $\deg(X)$  components and at least one of them has the same dimension  $\dim(X_i) = \dim(X)$ .

- (d) The sets  $X \cap Y$ ,  $\overline{X \cup Y}$ ,  $X \setminus \overline{Y}$  and  $X \times Y$  are also locally closed with the following bounds:

$$\begin{aligned} \dim(\overline{X \cup Y}) &= \max(\dim(X), \dim(Y)) \\ \deg(\overline{X \cup Y}) &\leq \deg(X) + \deg(Y) \\ \dim(X \cap Y) &\leq \min(\dim(X), \dim(Y)) \\ \deg(X \cap Y) &\leq \deg(X) \deg(Y) \\ \dim(X \setminus \overline{Y}) &\leq \dim(X) \\ \dim(X \times Y) &= \dim(X) + \dim(Y) \\ \deg(X \times Y) &= \deg(X) \deg(Y) \end{aligned}$$

Note that we cannot estimate  $\deg(X \setminus \overline{Y})$  in this generality.

- (e) Suppose that  $X$  is irreducible. Then each nonempty open subset  $U \subset X$  is dense in  $X$  with  $\dim(X \setminus U) < \dim(X)$  (and we do not bound the degree of  $X \setminus U$ ).
- (f) The direct product of irreducible algebraic sets is again irreducible.
- (g) If  $X$  is the common zero locus of degree  $d$  polynomials, then it is the common zero locus of at most  $(d+1)^m$  of them, and  $\deg(X) \leq d^m$ . On the other hand, a closed set  $X$  is the common zero locus of polynomials of degree at most  $\deg(X)$ .

Most of this Fact is proved in [27, Chapters I.1 and II.3]. The bound on  $\deg(X \cap Y)$  is (an appropriate version of) Bézout's theorem (see [23]) and (g) follows from [39, Section I.3].

We also need the following:

**Fact 20.** Let  $X$  and  $Y$  be affine algebraic sets and  $f : X \rightarrow Y$  a morphism. We define several (open, closed or locally closed) subsets of  $X$  and  $Y$ . Their dimension is at most  $\dim(X)$ , and we bound their degrees from above. We define the function  $\Phi(d) = (d+2)^{(d+1)^{\dim(X)+\deg(f)} 2^d}$  and the constant  $D = \Phi(\Phi(\dots \Phi(\deg(f)) \dots))^{\dim(X)+\deg(f)}$  where the function  $\Phi$  is iterated  $\dim(X) + \deg(f) - 1$  times.

- (a) There is a partition of  $\overline{f(X)}$  into at most  $D$  locally closed subsets  $Y_i$  of degree at most  $D$  such that the closure of each  $Y_i$  is the union of partition classes and either  $f^{-1}(Y_i) = \emptyset$  or  $\dim(f^{-1}(y)) = \dim(X) - \dim(Y_i)$  for all  $y \in Y_i$ .
- (b) We have  $\deg(\overline{f(X)}) \leq \deg(f)$ . The image  $f(X)$  contains a dense open subset of  $\overline{f(X)}$ . If  $X$  is irreducible then so is  $\overline{f(X)}$ .

- (c) For each  $y \in f(X)$  the fibre  $f^{-1}(y) \subseteq X$  is closed with  $\deg(f^{-1}(y)) \leq \deg(f)$ . For each closed set  $T \subseteq Y$  the subset  $f^{-1}(T)$  is also closed and its degree is at most  $\deg(T) \deg(f)$ .
- (d) The degree of the closed complement  $\overline{f(X)} \setminus f(X)$  is at most  $D^2$ .
- (e) Suppose that  $X$  is irreducible. For each  $t \in X$  we have

$$\dim(f^{-1}(f(t))) \geq \dim(X) - \dim(\overline{f(X)}) .$$

Those  $t \in X$  where equality holds form an open dense subset  $X_{\min} \subseteq X$  and  $\deg(X \setminus X_{\min}) \leq D^2 \deg(f)$ .

- (f) Let  $S \subseteq X$  be a closed subset that is the intersection of  $X$  and a closed set of degree  $d$ . Then the degree of the restricted morphism  $f|_S$  is at most  $d \cdot \deg(f)$ , hence  $\deg(\overline{f(S)}) \leq d \cdot \deg(f)$  (see (b)). If  $S$  is an irreducible component of  $X$  then there are better bounds:  $\deg(f|_S) \leq \deg(f)$  and  $\deg(\overline{f(S)}) \leq \deg(f)$ .

Parts (b), (c) and (f) as well as the fact that  $X_{\min}$  of (e) is open and dense follows easily using [27, Chapters I.1 and II.3] and Fact 19. Moreover, the closed complement considered in (d) is the union of a number of the locally closed subsets of (a), hence its degree bound follows immediately from (a). Similarly, the subset discussed in (e) is the inverse image of the union of a number of the locally closed subsets of (a), hence its degree is bounded by (a) and (c). So the only thing that remains to be proved is (a).

*Sketch of the proof of (a).* Let  $\overline{\mathbb{F}}^m \supseteq X$  and  $\overline{\mathbb{F}}^n \supseteq Y$  be the ambient affine spaces,  $\Gamma_f \subseteq \overline{\mathbb{F}}^m \times \overline{\mathbb{F}}^n$  the graph of  $f$ , and  $\pi : \overline{\mathbb{F}}^m \times \overline{\mathbb{F}}^n \rightarrow \overline{\mathbb{F}}^n$  the linear projection to the second factor. Then  $\Gamma_f$  is isomorphic to  $X$ , hence it is enough to find an analogous partition of  $\overline{\pi(\Gamma_f)} = \overline{f(X)}$  with respect to  $\pi$  and  $\Gamma_f$  (with the same bound  $D$  defined in terms of  $\deg(f)$  and  $\dim(X)$ ).

Let  $L$  denote the linear span of  $\Gamma_f$  and set  $\tilde{\pi} = \pi|_L$ . In general, for each variety  $V$  of degree at least 2, [27, Ex.I.7.7] constructs a cone containing  $V$  whose dimension is  $\dim(V) + 1$ , and whose degree is strictly smaller than  $\deg(V)$ . By iterating this cone-construction we arrive, in at most  $\dim(V) - 1$  steps, at a variety of degree 1. By [27, Ex.I.7.6] this iterated cone is a linear subspace, i.e. the original  $V$  is contained in a linear subspace of dimension at most  $\dim(V) + \deg(V) - 1$ . In particular, we have  $\dim(L) \leq \dim(\Gamma_f) + \deg(\Gamma_f) - 1 = \dim(X) + \deg(f) - 1$ . We need to find a partition of  $\overline{\tilde{\pi}(\Gamma_f)} = \overline{f(X)}$  as in (a) with respect to  $\tilde{\pi}$  and  $\Gamma_f$  (with the same bound  $D$ ). We factor  $\tilde{\pi}$  into  $\dim(L) - \dim(\tilde{\pi}(L)) \leq$

$\dim(X) + \deg(f) - 1$  consecutive linear projections  $\tilde{\pi}_j$ , each with one-dimensional fibres. Our strategy is the following. First we partition  $\overline{\tilde{\pi}_1(\Gamma_f)}$  via the next Claim 21. Then for each partition class  $C \subseteq \overline{\tilde{\pi}_1(\Gamma_f)}$  we apply again Claim 21, and partition the closed image  $\tilde{\pi}_2(\overline{C})$ . We obtain various partitions on partially overlapping subsets of  $\tilde{\pi}_2(\tilde{\pi}_1(\Gamma_f))$ . Let us consider the common refinement of them, it is a partition of  $\tilde{\pi}_2(\tilde{\pi}_1(\Gamma_f))$  into locally closed sets. We iterate this procedure, and obtain partitions of  $\tilde{\pi}_j \circ \cdots \circ \tilde{\pi}_1(\Gamma_f)$  for each  $j$ . (Note that  $k$  in these applications of Claim 21 is always at most  $\dim(X) + \deg(f) - 2$ .) In the last step we obtain a partition of  $\overline{\tilde{\pi}(\Gamma_f)} = \overline{f(X)}$  as required.  $\square$

**Claim 21.** *Let  $Z \subseteq \overline{\mathbb{F}}^k$  be a locally closed set and  $\Gamma$  be the common zero locus inside  $\overline{\mathbb{F}} \times Z$  of some polynomials of degree at most  $d$ .*

- (a) *Then  $Z$  has a partition into at most  $(d+2)^{(d+1)^{k+2}-1}$  locally closed subsets  $Z_i$  and there are corresponding  $(k+1)$ -variate polynomials  $P_i$  of degree at most  $d^{(d+1)^{k+1}2^d}$  such that*

$$\Gamma \cap (\overline{\mathbb{F}} \times Z_i) = \left\{ (t, \underline{z}) \in \overline{\mathbb{F}} \times Z_i \mid P_i(t, \underline{z}) = 0 \right\}$$

*for all  $i$ , and the closures  $\overline{Z_i}$  are defined via equations of degree at most  $d^{(d+1)^{k+1}2^d}$  plus the equations of  $\overline{Z}$ .*

- (b) *Those points  $\underline{z} \in Z_i$  for which  $\Gamma \cap (\overline{\mathbb{F}} \times \{\underline{z}\})$  has any prescribed number of points (it can be  $0, 1, \dots, d$  or  $\infty$ ) form a locally closed subset that is defined (inside  $Z$ ) via equations of degree at most  $d^{(d+1)^{k+1}2^d}$ , and the total number of these subsets is at most  $(d+2)^{(d+1)^{k+2}}$ .*
- (c) *Moreover, one may require both partitions to have the following additional property: the closure in  $Z$  of each partition class is the union of partition classes.*

*Sketch of proof.* The upper bounds and part (c) follow immediately from our construction, we leave them to the reader.  $\Gamma$  can be defined as the common zero locus inside  $\overline{\mathbb{F}} \times Z$  of at most  $(d+1)^{k+1}$  polynomials of degree at most  $d$  (see Fact 19.(g)). We prove (a) via induction on the number of defining polynomials. If  $\Gamma = \overline{\mathbb{F}} \times Z$  then there is nothing to prove. Otherwise let  $g$  be one of the nonzero defining polynomials of  $\Gamma$  and  $\Gamma' \subseteq \overline{\mathbb{F}} \times Z$  the common zero locus of the other defining polynomials. Applying the induction hypothesis to  $\Gamma'$  gives us a partition  $\bigcup_j Z'_j = Z$  and corresponding polynomials  $P'_j$ . Our goal is to refine this partition, i.e. find partitions  $Z'_j = \bigcup_i Z'_{ji}$  and find appropriate polynomials  $P'_{ji}$ . We shall find the  $Z'_{ji}$  one by one with the following algorithm.

The portion of  $\Gamma$  that lies inside  $\overline{\mathbb{F}} \times Z'_j$  is defined by the equations  $P'_j(t, \underline{z}) = g(t, \underline{z}) = 0$  (besides the equations and inequalities defining  $Z'_j$ ). We consider  $g$  and  $P'_j$  as polynomials in the variable  $t$  whose coefficients are polynomial functions of the parameter  $\underline{z}$ . Note that  $g$  and  $P'_j$  as well as all the polynomials  $P'_{ji}$  we construct below have  $t$ -degrees at most  $d$ . Our plan is to find the gcd of  $g$  and  $P'_j$  with respect to the variable  $t$  for all values of  $\underline{z}$  simultaneously. In order to do so we try to run Euclid's algorithm simultaneously for all  $\underline{z}$ . There are two obstacles we have to overcome. First, for different values of  $\underline{z}$  the algorithm needs a different number of steps to complete. Second, to do a polynomial division uniformly for several values of  $\underline{z}$  we have to make sure that the degree of the divisor do not vary with  $\underline{z}$  (i.e. we can talk about the leading coefficient). So before each polynomial division we construct also a partition of  $Z'_j$ , always refining the partition obtained in the previous step, so that the upcoming division can be done uniformly for values  $\underline{z}$  lying in the same partition class.

To begin with, let  $Z'_{j0}$  and  $Z'_{j1}$  denote the loci of those  $\underline{z} \in Z'_j$  where all coefficients of  $g$  or  $P'_j$  respectively vanish. We set  $P'_{j0} = P'_j$  and  $P'_{j1} = g$ . Similarly, for each pair of integers  $0 \leq a, b \leq d$  we consider the locus of those  $\underline{z} \in Z'_j$  where the  $t$ -degrees of  $g$  and  $P'_j$  are just  $a$  and  $b$ . This is a partition of  $Z'_j$  into locally closed subsets, each defined via the vanishing or non-vanishing of a number of coefficients. For parameter values  $\underline{z}$  lying in  $Z'_{j0}$  or  $Z'_{j1}$  the algorithm stops right away with gcd equal to  $P'_{j0}$  or  $P'_{j1}$ . On the other hand, for any other partition class  $\tilde{Z} \subseteq Z'_j$  we can do the first polynomial division uniformly for all  $\underline{z} \in \tilde{Z}$ .

During the algorithm we do similar subdivisions again and again. Suppose that we completed a number of polynomial divisions and constructed the partition corresponding to the last completed division. Let  $\tilde{Z}$  be a class of that partition and suppose that the algorithm is still running for  $\underline{z} \in \tilde{Z}$  and  $\tilde{g}$  and  $\tilde{r}$  are the divisor and the remainder of the last completed polynomial division for all values  $\underline{z} \in \tilde{Z}$ . We consider the locus of those  $\underline{z} \in \tilde{Z}$  where all coefficients of  $\tilde{r}$  vanish (here  $\tilde{g}$  does not vanishes). This will be our next  $Z'_{ji}$  (whatever  $i$  follows now). For  $\underline{z} \in Z'_{ji}$  Euclid's algorithm stops at this stage, and we set  $P'_{ji} = \tilde{g}$ , the gcd we obtain. As before, we partition  $\tilde{Z} \setminus Z'_{ji}$  according to the  $t$ -degree of  $\tilde{r}$  (here the  $t$ -degree of  $\tilde{g}$  is unimportant). Then we can do the polynomial division  $\tilde{g} : \tilde{r}$  uniformly for values  $\underline{z}$  lying in the same partition class. This way we obtain our new remainders (one for each partition class), and Euclid's algorithm continues.

It is clear that for each  $\underline{z} \in Z'_j$  the gcd is found in at most  $\deg(g)+1 \leq d+1$  steps, hence we obtain the promised partition  $Z'_j = \cup_i Z'_{ji}$ . The induction step is complete.

Part (b) follows from part (a). Indeed, the portion of  $\Gamma$  that lies inside  $\overline{\mathbb{F}} \times Z_i$  is defined by the equation  $P_i(t, \underline{z}) = 0$  (besides the equations of  $Z_i$ ). For each  $\underline{z} \in Z_i$  the number of points in  $\Gamma \cap (\overline{\mathbb{F}} \times \{\underline{z}\})$  is either  $\infty$  (in case all  $t$ -coefficients of  $P_i$  are zero at  $\underline{z}$ ), or equal to the  $t$ -degree of the polynomial  $P_i(t, \underline{z})$  (which is at most  $d$ ). The locus of those  $\underline{z}$  which correspond to a given degree can be defined via the vanishing or nonvanishing of a number of  $t$ -coefficients of  $P_i(t, \underline{z})$ . This proves the claim.  $\square$

#### 4. CONCENTRATION IN GENERAL

Let  $\alpha \subseteq \overline{\mathbb{F}}^m$  be a finite subset. An essential part of our general strategy is to find closed sets  $X$  which contain a large number of elements of  $\alpha$  compared to their dimension. To measure the relative size of  $\alpha \cap X$  we introduce the following:

**Definition 22.** For each subset  $X \subseteq \overline{\mathbb{F}}^m$  with  $\dim(\overline{X}) > 0$  we define the *concentration* of  $\alpha$  in  $X$  as follows:

$$\mu(\alpha, X) = \frac{\log |\alpha \cap X|}{\dim(\overline{X})}$$

For simplicity, here and everywhere in this paper,  $\log$  stands for the natural logarithm. When  $\alpha \cap X = \emptyset$ , we set  $\mu(\alpha, X) = -\infty$ .

In this section we first show that the concentration in a closed set  $X$  does not decrease too much when we take an appropriate irreducible closed subset.

**Proposition 23.** *Let  $X \subseteq Y \subseteq \overline{\mathbb{F}}^m$  be closed sets of positive dimension. Then for all finite sets  $\alpha \subseteq \beta \subseteq \overline{\mathbb{F}}^m$  with  $\alpha \cap X \neq \emptyset$  we have:*

$$(1) \quad 0 \leq \mu(\alpha, X) \leq \mu(\beta, X) \leq \frac{\dim(Y)}{\dim(X)} \cdot \mu(\beta, Y)$$

and for all integers  $n > 0$  the  $n$ -fold direct products satisfy

$$(2) \quad \mu\left(\prod^n \alpha, \prod^n X\right) = \mu(\alpha, X) .$$

*Proof.* Clear from the definition.  $\square$

**Lemma 24.** *Let  $Z \subseteq \overline{\mathbb{F}}^m$  be a closed set with  $\dim(Z) > 0$  and  $\alpha \subseteq \overline{\mathbb{F}}^m$  a finite subset with  $|\alpha \cap Z| > \deg(Z)$ . Then there is an irreducible component  $Z' \subseteq Z$  such that  $\dim(Z') > 0$  and*

$$(3) \quad \mu(\alpha, Z') \geq \mu(\alpha, Z) - \log(\deg(Z)) .$$

*Proof.* Since  $Z$  has at most  $\deg(Z)$  irreducible components (see Fact 19.(c)) there is a component  $Z' \subseteq Z$  with

$$(4) \quad |\alpha \cap Z'| \geq \frac{|\alpha \cap Z|}{\deg(Z)} > 1 .$$

In particular we have  $\dim(Z') > 0$ . We take the logarithm of inequality (4), divide the two sides by  $\dim(Z')$  and rewrite it in terms of concentrations. Using  $\dim(Z') \leq \dim(Z)$  we obtain

$$\begin{aligned} \mu(\alpha, Z') &\geq \frac{\dim(Z)}{\dim(Z')} \mu(\alpha, Z) - \frac{\log(\deg(Z))}{\dim(Z')} \geq \\ &\geq \mu(\alpha, Z) - \log(\deg(Z)) \end{aligned}$$

as required.  $\square$

The proof of Lemma 24 involves a choice. For proving Theorem 6 it will be important to use constructions that are uniquely determined. To this end we order the finite set  $\alpha$ , and use this order to make the choices unique. Of course,  $\alpha$ -valued sequences and subsets of  $\alpha$  can be ordered lexicographically.

In the rest of the paper we state several existence results. However, in the proofs we typically use explicit constructions. When we write that our construction of a subset (or a tuple of elements, etc.) is uniquely determined, we understand that the result of the construction depends uniquely on the input data (which usually involves an ordered set  $\alpha$ ).

**Lemma 25.** *For all  $N > 0$  and  $\Delta > 0$  there are reals  $B = B_{\text{irr}}(N, \Delta) \geq 0$  and  $K = K_{\text{irr}}(N, \Delta) \geq 0$  with the following property.*

*Let  $Z \subseteq \overline{\mathbb{F}}^m$  be a closed set and  $\alpha \subseteq \overline{\mathbb{F}}^m$  an ordered finite subset. Suppose that  $0 < \dim(Z) \leq N$ ,  $\deg(Z) \leq \Delta$  and  $|\alpha \cap Z| \geq K$ . Then there is an irreducible closed subset  $Z' \subseteq Z$  such that  $\dim(Z') > 0$ ,  $\deg(Z') \leq B$  and*

$$\mu(\alpha, Z') \geq \mu(\alpha, Z) - \log(B) .$$

*Moreover, our construction of  $Z'$  is uniquely determined.*

*Proof.* Let  $B = \Delta^{(N+1)^N}$  and set  $K > \Delta^{2N(N+1)^N}$ . Then

$$(5) \quad \mu(\alpha, Z) \geq \frac{\log(K)}{N} > \log(\Delta^{2(N+1)^N}) .$$



We build by induction a sequence  $Z = Z_0 \supset Z_1 \supset Z_2 \supset \cdots \supset Z_I$  of closed subsets such that

$$(6) \quad \begin{aligned} 0 &< \dim(Z_{i+1}) < \dim(Z_i) , \\ \deg(Z_{i+1}) &\leq \deg(Z_i)^{N+1} \leq \Delta^{(N+1)^{i+1}} , \\ \mu(\alpha, Z_i) &\geq \mu(\alpha, Z) - \log \left( \Delta^{i(N+1)^{i-1}} \right) . \end{aligned}$$

for all  $0 \leq i < I$ . Since the dimensions are strictly decreasing, such a sequence has length  $I+1 \leq N$ . Suppose  $Z_i$  is already constructed. If it is irreducible, we stop the induction and set  $Z' = Z_i$ , the lemma holds in this case. Otherwise, it follows from (5) and (6) that  $|\alpha \cap Z_i| > \Delta^{(N+1)^N} > \deg(Z_i)$  and we may apply Lemma 24. So there is an irreducible component  $Z'_i \subseteq Z_i$  such that  $\dim(Z'_i) > 0$  and

$$(7) \quad \mu(\alpha, Z'_i) \geq \mu(\alpha, Z_i) - \log(\deg(Z_i)) \geq \mu(\alpha, Z) - \log(\Delta^{(i+1)(N+1)^i}) .$$

Of course, there are possibly many choices for  $Z'_i$ , we choose one in such a way that the subset  $\alpha_i = \alpha \cap Z'_i$  is lexicographically minimal among the possible intersections. Note that  $\alpha_i$  is uniquely determined, but  $Z'_i$  may not be. Then  $\mu(\alpha_i, Z'_i) = \mu(\alpha, Z'_i)$  and using (5) and (7) we obtain  $|\alpha_i| > \deg(Z_i)^{N+1}$ . If  $Z'_i$  is the only irreducible component containing  $\alpha_i$  then it is uniquely determined. We stop the induction and set  $Z' = Z'_i$ , the lemma holds in this case.

Otherwise let  $T_1, T_2, \dots$  denote those irreducible components of  $Z_i$  which contain  $\alpha_i$  and let  $Z_{i+1} = \bigcap^j T_j$  be their intersection, this is again uniquely determined. Clearly  $\dim(Z_{i+1}) < \dim(Z_i)$  and we shall prove that

$$\deg(Z_{i+1}) \leq \deg(Z_i)^{N+1} .$$

In fact it is more convenient to prove a slightly stronger statement: for each closed subset  $W \subseteq Z_i$  we have

$$(8) \quad \deg(W \cap Z_{i+1}) \leq \deg(W) \cdot \deg(Z_i)^{\dim(W)} .$$

We prove (8) by induction on  $\dim(W)$ , it obviously holds for  $\dim(W) = 0$ . Assume for a moment that  $W$  is irreducible. If it is contained in all  $T_j$  then  $W \cap Z_{i+1} = W$  and (8) holds. On the other hand, if say  $W \not\subseteq T_1$  then  $W' = W \cap T_1$  has smaller dimension, hence satisfies the analogue of (8). But  $\deg(W') \leq \deg(W) \deg(T_1) \leq \deg(W) \deg(Z_i)$ , so we have

$$\begin{aligned} \deg(W \cap Z_{i+1}) &= \deg(W' \cap Z_{i+1}) \leq \\ &\leq \deg(W') \deg(Z_i)^{\dim(W)-1} \leq \deg(W) \deg(Z_i)^{\dim(W)} . \end{aligned}$$

as we promised. In order to complete the induction step for a reducible  $W$  we simply add up the analogous inequalities for each component of  $W$ .

Then  $\dim(Z_{i+1}) > 0$  by Remark 16. Now we have

$$\mu(\alpha, Z_{i+1}) = \mu(\alpha_i, Z_{i+1}) > \mu(\alpha_i, Z'_i) = \mu(\alpha, Z'_i),$$

hence  $Z_{i+1}$  satisfies (6). As we noted earlier, the induction must stop in at most  $N$  steps, which proves the lemma.  $\square$

Next we show that the concentration in a closed set  $X$  does not decrease too much when we map  $X$  somewhere by a “nice” morphism.

**Lemma 26.** *Let  $Z \subseteq \overline{\mathbb{F}}^m$  be an irreducible closed set,  $\alpha \subset \overline{\mathbb{F}}^m$  an ordered nonempty finite set and  $f : Z \rightarrow \overline{\mathbb{F}}^l$  a morphism such that*

$$\dim(Z) > \dim(\overline{f(Z)}) > 0$$

and

$$\dim(Z) = \dim(\overline{f(Z)}) + \dim(f^{-1}(t))$$

for all  $t \in f(\alpha \cap Z)$ . Then there is a fibre  $S = f^{-1}(s)$ ,  $s \in f(\alpha \cap Z)$  such that for each value (negative, positive or 0) of the parameter  $\varepsilon$  one has

$$(9) \quad \begin{cases} \text{either} & \mu(f(\alpha \cap Z), \overline{f(Z)}) \geq \mu(\alpha, Z) - \varepsilon \dim(S) \\ \text{or} & \mu(\alpha, S) \geq \mu(\alpha, Z) + \varepsilon \dim(\overline{f(Z)}) \end{cases}$$

Moreover, our construction of  $S$  is uniquely determined.

Note that if all nonempty fibres of  $f$  have the same dimension, then the condition  $\dim(Z) = \dim(\overline{f(Z)}) + \dim(f^{-1}(t))$  is satisfied (see Fact 20.(e)). Note also that  $S$  is a closed set with  $\deg(S) \leq \deg(f)$  by Fact 20.(c).

*Proof.* Let us consider those fibres  $f^{-1}(t)$  where the number of points  $|\alpha \cap f^{-1}(t)|$  is maximal, and let  $S = f^{-1}(s)$  be the one among them for which the subset  $\alpha \cap S \subseteq \alpha$  is lexicographically minimal. Then by assumption we have

$$0 < \dim(S) = \dim(Z) - \dim(\overline{f(Z)}) < \dim(Z).$$

We have

$$|\alpha \cap Z| = \sum_{t \in f(\alpha \cap Z)} |\alpha \cap f^{-1}(t)|,$$

hence

$$|\alpha \cap Z| \leq |f(\alpha \cap Z)| \cdot |\alpha \cap S|$$

We take the logarithm of our inequality and rewrite it in terms of concentrations:

$$\mu(\alpha, Z) \cdot \dim(Z) \leq \mu(f(\alpha \cap Z), \overline{f(Z)}) \cdot \dim(\overline{f(Z)}) + \mu(\alpha, S) \cdot \dim(S)$$

We divide both sides by  $\dim(Z)$  and we introduce two extra terms involving  $\varepsilon$  on the right hand side which cancel each other:

$$\mu(\alpha, Z) \leq \left[ \mu(f(\alpha \cap Z), \overline{f(Z)}) + \varepsilon \dim(S) \right] \frac{\dim(\overline{f(Z)})}{\dim(Z)} + \left[ \mu(\alpha, S) - \varepsilon \dim(\overline{f(Z)}) \right] \frac{\dim(S)}{\dim(Z)}$$

On the right hand side we see a weighted arithmetic mean of the two expressions in square brackets. We can certainly bound it from above with the larger of them, which justifies our statement.  $\square$

The following extension of Lemma 26 is our basic tool for transporting large concentration from one subset to another. The idea is that if the transport fails then we get an even larger concentration somewhere inside the first subset.

**Lemma 27** (Transport). *For all  $\Delta > 0$  there is a real  $B = B_{\text{transport}}(\Delta) \geq 0$  with the following property. Let  $X$  be an affine algebraic set,  $Z \subseteq X$  a closed subset and  $f : X \rightarrow \overline{\mathbb{F}}^m$  be a morphism with  $\deg(Z) \leq \Delta$ ,  $\deg(f) \leq \Delta$  and  $\dim(\overline{f(Z)}) > 0$ . Suppose that  $Z$  is irreducible. Then for all ordered finite subsets  $\alpha \subseteq X$  and all  $\varepsilon \geq 0$  either*

$$(10) \quad \mu(f(\alpha), \overline{f(Z)}) \geq \mu(\alpha, Z) - \log(B) - \varepsilon \cdot \dim(Z)$$

*or there is a closed subset  $S \subset Z$  such that  $\deg(S) \leq B$ ,  $0 < \dim(S) < \dim(Z)$  and*

$$(11) \quad \mu(\alpha, S) \geq \mu(\alpha, Z) - \log(B) + \varepsilon.$$

*Moreover, our construction of  $S$  is uniquely determined.*

Note, that the condition  $\dim(\overline{f(Z)}) > 0$  implies that  $\dim(Z) > 0$ , hence the concentrations appearing in the lemma are defined.

*Proof.* To simplify notation we replace  $\alpha$  with  $\alpha \cap Z$ ,  $X$  with  $Z$ ,  $\Delta$  with  $\Delta^2$  (see Fact 20.(f)) and  $f$  with its restriction to  $Z$ , then  $\alpha \subseteq Z$ . If  $\alpha = \emptyset$  then (10) holds automatically since the right hand side is  $-\infty$ . So we assume  $\alpha \neq \emptyset$ . This implies that  $f(\alpha) \neq \emptyset$ , hence the left hand side of (10) is non-negative. If  $\mu(\alpha, Z) \leq \log(B)$  then inequality (10) obviously holds since the right hand side is nonpositive. So we assume  $\mu(\alpha, Z) > \log(B)$  which implies  $|\alpha| > B$ .

First we prove a special case:

$$(12) \quad \begin{array}{l} \text{If } \dim(f^{-1}(t)) = \dim(Z) - \dim(\overline{f(Z)}) \text{ for all } t \in f(\alpha) \\ \text{then the lemma is true with any } B \geq 1 + \Delta. \end{array}$$

If  $\dim(Z) > \dim(\overline{f(Z)})$  then we apply Lemma 26 with parameter  $\varepsilon$ . We get a fibre  $S = f^{-1}(s)$  satisfying (9). Since  $\varepsilon \geq 0$ , we may replace  $\varepsilon \dim(\overline{f(Z)})$  with  $\varepsilon$  and  $\varepsilon \dim(S)$  with  $\varepsilon \dim(Z)$ , hence either (10) or

(11) holds for any  $B \geq 1$ . By Fact 20.(c)  $S = f^{-1}(s)$  is closed and  $\deg(S) \leq \Delta$ , hence (12) is proved in this case.

On the other hand, if  $\dim(Z) = \dim(\overline{f(Z)})$  (and we are still in the special case of (12)), then all points of  $\alpha$  are contained in finite fibres of  $f$ , and the number of points in each finite fibre is at most  $\deg(f) \leq \Delta$  (see Fact 20.(c)). Hence

$$\mu(f(\alpha), \overline{f(Z)}) = \frac{\log |f(\alpha)|}{\dim(\overline{f(Z)})} \geq \frac{\log(|\alpha|/\Delta)}{\dim(Z)} \geq \mu(\alpha, Z) - \log(\Delta),$$

and therefore (10) holds for any  $B \geq \Delta$ . The special case (12) is proved.

Next we prove the lemma in full generality. We define the following subset:

$$\alpha' = \left\{ z \in \alpha \mid \dim(f^{-1}(f(z))) = \dim(Z) - \dim(\overline{f(Z)}) \right\}.$$

First we deal with the case  $|\alpha'| \geq |\alpha|/2$ . We have

$$\mu(\alpha', Z) = \frac{\log |\alpha'|}{\dim(Z)} \geq \frac{\log |\alpha| - \log(2)}{\dim(Z)} \geq \mu(\alpha, Z) - \log(2).$$

We apply the special case (12) of the lemma to  $\alpha'$  and  $Z$ . We obtain that either

$$\begin{aligned} \mu(f(\alpha), \overline{f(Z)}) &\geq \mu(f(\alpha'), \overline{f(Z)}) \geq \\ &\geq \mu(\alpha', Z) - \log(1 + \Delta) - \varepsilon \cdot \dim(Z) \geq \\ &\geq \mu(\alpha, Z) - \log(2 + 2\Delta) - \varepsilon \cdot \dim(Z), \end{aligned}$$

or there is a closed subset  $S \subset Z$  such that  $\deg(S) \leq 1 + \Delta$ ,  $0 < \dim(S) < \dim(Z)$  and

$$\begin{aligned} \mu(\alpha, S) &\geq \mu(\alpha', S) \geq \mu(\alpha', Z) - \log(1 + \Delta) + \varepsilon \geq \\ &\geq \mu(\alpha, Z) - \log(2 + 2\Delta) + \varepsilon. \end{aligned}$$

The lemma holds in this case with any  $B \geq 2 + 2\Delta$ .

In the remaining case we have  $|\alpha'| < |\alpha|/2$ . Setting

$$S = \left\{ z \in Z \mid \dim(f^{-1}(f(z))) > \dim(Z) - \dim(\overline{f(Z)}) \right\}$$

we have  $|\alpha \cap S| > \frac{1}{2}|\alpha|$ .

The irreducibility of  $Z$  implies (see Fact 20.(e) and Fact 19.(e)) that  $S$  is a closed subset of  $Z$  and  $\dim(S) < \dim(Z)$ ,  $\deg(S) \leq \Delta'$  with a certain bound  $\Delta' = \Delta'(\dim(Z), \Delta)$ . We set

$$B = B_{\text{transport}}(\Delta) = \max(2 + 2\Delta, 2\Delta').$$

Then the set  $S$  has at least  $|\alpha \cap S| > |\alpha|/2 \geq B/2 \geq \Delta'$  points, hence  $\dim(S) > 0$  (see Remark 16). Therefore  $\mu(\alpha, S)$  is defined and we can write:

$$\begin{aligned} \mu(\alpha, S) &= \frac{\log |\alpha \cap S|}{\dim(S)} \geq \frac{\log |\alpha| - \log(2)}{\dim(S)} \geq \\ &\geq \frac{\dim(Z)}{\dim(S)} \mu(\alpha, Z) - \log(2) \geq \mu(\alpha, Z) - \log(B) + \frac{\mu(\alpha, Z)}{\dim(S)}. \end{aligned}$$

We compare now the last term to  $\varepsilon$ . If  $\varepsilon \leq \frac{\mu(\alpha, Z)}{\dim(S)}$  then inequality (11) holds. On the other hand, for  $\varepsilon > \frac{\mu(\alpha, Z)}{\dim(S)} \geq \frac{\mu(\alpha, Z)}{\dim(Z)}$  the inequality (10) holds, since its right hand side becomes negative. We proved the lemma in all cases.  $\square$

## 5. CLOSED SETS IN GROUPS

**Definition 28.** A *linear algebraic group* is a closed subgroup  $G \leq GL(n, \overline{\mathbb{F}})$ . We use this matrix realisation of  $G$  to calculate degrees of closed subsets. We shall denote by  $\text{mult}(G)$  and  $\text{inv}(G)$  the degrees of the morphisms  $(g, h) \rightarrow gh$  and  $g \rightarrow g^{-1}$ .

As usual,  $\mathcal{Z}(G)$ ,  $[G, G]$  and  $G^0$  denote the centre, the commutator subgroup and the unit component of  $G$ , and for any subset  $A \subseteq G$  we denote by  $\langle A \rangle$ ,  $\mathcal{N}_G(A)$  and  $\mathcal{C}_G(A)$  the generated subgroup, the normaliser and the centraliser of  $A$ . The subgroup  $\mathcal{C}_G(A)^0$  is usually called the *connected centraliser* of  $A$ . We shall often use products of several elements and subsets in the usual sense. In order to distinguish from this kind of product, the  $m$ -fold direct product of a subset  $\alpha \subseteq G$  is denoted by  $\prod^m \alpha \subseteq \prod^m G$ .

**Definition 29.** Let  $\alpha \subseteq GL(n, \overline{\mathbb{F}})$  be an ordered finite subset. This ordering extends to an ordering of the subgroup  $\langle \alpha \rangle$  (hence to  $\alpha^i$  for all  $i$ ) in a natural way. We shall use this extension without further reference.

*Remark 30.* We measure the complexity of a closed subset  $X \subseteq \overline{\mathbb{F}}^m$  with two numerical invariants:  $\dim(X)$  and  $\deg(X)$ . In contrast, we measure the complexity of a closed subgroup  $G \leq GL(n, \overline{\mathbb{F}})$  with four numerical invariants:  $\dim(G)$ ,  $\deg(G)$ ,  $\text{mult}(G)$  and  $\text{inv}(G)$ . In order to reduce the number of variables to two, say  $N$  and  $\Delta$ , we shall consider groups  $G$  with  $\dim(G) \leq N$ ,  $\deg(G) \leq \Delta$ ,  $\text{mult}(G) \leq \Delta$  and  $\text{inv}(G) \leq \Delta$ .

It can be tiresome to bound all four numerical invariants of  $G$ . By the following proposition in most cases it is enough to bound only  $\dim(G)$  and  $\deg(G)$ .

**Proposition 31.** *Let  $G$  be a linear algebraic group and  $H \leq G$  a closed subgroup. Then  $\text{mult}(H) \leq \deg(H)^2 \cdot \text{mult}(G)$  and  $\text{inv}(H) \leq \deg(H) \cdot \text{inv}(G)$ . In particular, if  $G = GL(n, \overline{\mathbb{F}})$  then we have  $\text{mult}(H) \leq \deg(H)^2 \cdot 2^{n^2}$  and  $\text{inv}(H) \leq \deg(H) \cdot (n+1)^{n^2}$ .*

*Proof.* Follows immediately from Fact 20.(f) and Fact 19.(d).  $\square$

**Fact 32.** *Let  $G$  be a linear algebraic group. Suppose that  $f : \prod^m G \rightarrow \prod^n G$  is a morphism for some integers  $m, n > 0$  whose  $n$  coordinates are all defined to be product expressions (evaluated in the group  $G$ ) of length at most  $k$  of some fixed group elements, the  $m$  variables and their inverses. Then  $\deg(\overline{f(G)}) \leq \deg(f) \leq \text{inv}(G)^l \text{mult}(G)^{n(k-1)}$  where  $l \leq nk$  denotes the total number of times inverted variables occur in the  $n$  expressions (see Fact 20.(b)). If the product expressions do not contain the inverse of the variables then of course the bound does not depend on  $\text{inv}(G)$ .*

**Definition 33.** Let  $G$  be a linear algebraic group. For all  $m > 0$  and for each sequence  $\underline{g} = (g_1, g_2, \dots, g_m)$ ,  $g_i \in G$  we define the morphism

$$\tau_{\underline{g}} : \prod^m G \rightarrow G ,$$

$$\tau_{\underline{g}}(a_1, \dots, a_m) = (g_1^{-1} a_1 g_1)(g_2^{-1} a_2 g_2) \dots (g_m^{-1} a_m g_m) ,$$

*Remark 34.* Let  $G$  be a linear algebraic group and  $\underline{g} = (g_1, g_2, \dots, g_m)$  any sequence. Suppose that  $\dim(G) \leq N$ ,  $\deg(G) \leq \Delta$  and  $\text{mult}(G) \leq \Delta$  for certain values  $N$  and  $\Delta$ . According to Fact 32 there is a common upper bound on the degrees:

$$\deg(\tau_{\underline{g}}) \leq \Delta_{\tau}(m, N, \Delta) .$$

In fact, it is easy to see that conjugation by  $g_i$  is a linear transformation hence  $\deg(\tau_{\underline{g}}) \leq \text{mult}(G)^{m-1} \leq \Delta^{m-1}$ .

**Fact 35.** *Let  $G$  be a connected linear algebraic group and  $A, B \subseteq G$  arbitrary subsets. Then*

$$AB \subseteq \overline{A} \overline{B} \subseteq \overline{AB} .$$

We give a short proof, see also [34, page 56]. Let us consider the multiplication map  $f : G \times G \rightarrow G$ . If  $AB = f(A \times B)$  satisfies a polynomial equation  $p = 0$  then  $p(f(A \times B)) = 0$ , i.e. the polynomial  $p(f(-))$  vanishes on  $A \times B$ . But then it must vanish on its closure  $\overline{A \times B} = \overline{A} \times \overline{B}$ , hence  $p$  vanishes on  $f(\overline{A} \times \overline{B}) = \overline{A} \overline{B}$ .  $\square$

Closed subgroups of an algebraic group can be very complicated. In contrast, centraliser subgroups are defined by linear equations, and normalisers of a closed subset  $X$  can be defined in terms of the equations of  $X$ . This proves that

**Fact 36.** *Let  $G$  be a linear algebraic group.*

- (a) *The centraliser  $\mathcal{C}_G(X)$  of any subset  $X \subseteq G$  is closed and its numerical invariants are bounded:  $\deg(\mathcal{C}_G(X)) \leq \deg(G)$ ,  $\text{mult}(\mathcal{C}_G(X)) \leq \text{mult}(G)$  and  $\text{inv}(\mathcal{C}_G(X)) \leq \text{inv}(G)$ . If  $X$  is closed then its normaliser  $\mathcal{N}_G(X)$  is also closed and its numerical invariants are also bounded:  $\deg(\mathcal{N}_G(X)) \leq \deg(G) \deg(X)^{\dim(G)}$ ,  $\text{mult}(\mathcal{N}_G(X)) \leq \text{mult}(G) \deg(X)^{\dim(G)}$  and  $\text{inv}(\mathcal{N}_G(X)) \leq \text{inv}(G) \deg(X)^{\dim(G)}$ .*
- (b) *Cosets of a closed subgroup  $H \leq G$  are also closed, they all have the same degree. Therefore*

$$|G : G^0| = \frac{\deg(G)}{\deg(G^0)} \leq \deg(G).$$

Later we plan to apply the Transport Lemma 27 to various morphisms of the form  $\tau_{\underline{g}}$ . In the rest of this section we construct the appropriate sequences  $\underline{g}$ .

The following proposition gives a morphism which maps a direct power of a given closed subset  $Y$  onto a closed subgroup  $H$ . It should be considered folklore, see e.g. [34, Proposition on page 55] for a similar statement. Nevertheless, for the sake of completeness, we include a proof.

**Proposition 37.** *Let  $Y \subseteq GL(n, \overline{\mathbb{F}})$  be an irreducible closed subset of positive dimension and  $1 \in \alpha \subset GL(n, \overline{\mathbb{F}})$  an ordered finite subset. Let  $H \leq GL(n, \overline{\mathbb{F}})$  denote the smallest closed subgroup which is normalised by  $\alpha$  and contains  $Y$ . Suppose that  $\dim(H) \leq m$ . Then there is a sequence  $\underline{g} = (g_1, g_2, \dots, g_{2m})$  of elements  $g_i \in \alpha^{m-1}$  such that*

$$H = \tau_{\underline{g}} \left( \prod^{2m} (Y^{-1}Y) \right) = (g_1^{-1}Y^{-1}Yg_1)(g_2^{-1}Y^{-1}Yg_2) \dots (g_{2m}^{-1}Y^{-1}Yg_{2m}).$$

*Moreover, our construction of  $\underline{g}$  is uniquely determined,  $H$  is connected and there is a universal bound  $\deg(H) \leq \delta(m, \deg(\overline{Y^{-1}Y}))$ .*

*Remark 38.* In applications the dimension of  $H$  may not be known, but if  $G \leq GL(n, \overline{\mathbb{F}})$  is any closed subgroup normalised by  $\alpha$  which contains  $Y$  then one may set  $m = \dim(G)$  and one may also use the bound

$$\deg(\overline{Y^{-1}Y}) \leq \text{inv}(G) \cdot \text{mult}(G) \cdot \deg(Y)^2$$

(see Fact 19.(d) and Fact 20.(f)).

*Proof.* We set  $g_1 = 1$ . We will define  $g_i \in \alpha^{i-1}$  by induction and consider the product sets

$$Z_i = (g_1^{-1}Y^{-1}Yg_1)(g_2^{-1}Y^{-1}Yg_2) \dots (g_i^{-1}Y^{-1}Yg_i) \subseteq H.$$

Suppose that  $g_1, g_2, \dots, g_i$  are already defined. We set  $g_{i+1} \in \alpha^i$  to be the first element such that

$$\dim(\overline{Z_i}) < \dim\left(\overline{Z_i \cdot (g_{i+1}^{-1} Y^{-1} Y g_{i+1})}\right),$$

if there is any. Since the dimension of  $\overline{Z_i}$  is strictly increasing, eventually we must arrive to an index  $i \leq m$  so that  $g_{i+1}$  does not exist. But then for all  $g \in \alpha^i$  the closed subsets

$$\overline{Z_i} \subseteq \overline{Z_i \cdot (g^{-1} Y^{-1} Y g)}$$

are irreducible (see Fact 20.(b)) of the same dimension, hence they are equal. This implies that  $\overline{Z_i}^2 \subseteq \overline{Z_i}$  and  $g^{-1} \overline{Z_i} g \subseteq \overline{Z_i}$  for all  $g \in \alpha$ , hence  $\overline{Z_i}$  is a closed connected subgroup normalised by  $\alpha$  i.e.  $\overline{Z_i} = H$ . By Fact 20.(b) the product  $Z_i$  contains a dense open subset of  $H$ , hence  $H = Z_i^2$  by [34, Lemma on page 54]. Setting  $g_{i+j} = g_j$  for  $1 \leq j \leq i$  and  $g_{2i+1} = \dots g_{2m} = 1$  we obtain our statement.  $\square$

**Lemma 39.** *Let  $G \leq GL(n, \overline{\mathbb{F}})$  be a closed subgroup,  $Z \subseteq G \times G$  an irreducible closed set and  $(a, b) \in Z$ . Suppose that  $\tau_{(1,1)}(\overline{Z})$  has dimension 0 i.e. it is a finite set. Then there is an irreducible closed subset  $A \subseteq G$  such that*

$$(13) \quad Z = \left\{ (ah, h^{-1}b) \mid h \in A \right\}$$

and

$$\left\{ c \in GL(n, \overline{\mathbb{F}}) \mid \dim\left(\overline{\tau_{(c,1)}(Z)}\right) = 0 \right\} = \mathcal{C}_{GL(n, \overline{\mathbb{F}})}(A).$$

Note that in the proof we define  $A$  explicitly (hence uniquely), but we do not use this fact later.

*Remark 40.* Equation (13) implies immediately that  $\dim(A) = \dim(Z)$  and  $1 \in A$ .

*Proof.* By assumption  $\tau_{(1,1)}(Z)$  is finite and its closure is irreducible (see Fact 20.(b)), hence it is the single point  $ab \in G$ . Let  $\text{pr}_1 : G \times G \rightarrow G$  denote the projection on the first factor. We set

$$A = a^{-1} \text{pr}_1(Z).$$

We shall prove later, that it is in fact closed. Anyway,  $\overline{A}$  is irreducible (see Fact 20.(b)) and by definition  $1 = a^{-1}a \in A$ . Then each point of  $Z$  has the form  $(ah, \beta)$  with some  $h \in A$  and  $\beta \in G$ , and for all  $h \in A$  there must exist at least one such point. But then  $ab = \tau_{(1,1)}(ah, \beta) = ah\beta$  hence  $\beta = h^{-1}b$ . This proves equation (13). The set  $Z$  is closed, hence  $A$  is closed by equation (13). Now

$$\tau_{(c,1)}(Z) = \left\{ c^{-1}(ah)c(h^{-1}b) \mid h \in A \right\} = c^{-1}a \left\{ hch^{-1} \mid h \in A \right\} b$$



for all  $c \in GL(n, \overline{\mathbb{F}})$ . This has dimension 0 iff the set  $\{hch^{-1} | h \in A\}$  is finite. But  $A$  is irreducible, hence its closed image  $\overline{\{hch^{-1} | h \in A\}}$  is also irreducible (see Fact 20.(b)), so it is finite iff it is a single point (see Fact 16) i.e. iff  $hch^{-1}$  is independent of  $h \in A$ . But  $1 \in A$ , hence this last condition is equivalent to  $hch^{-1} = c$  for all  $h \in A$ , which simply means that  $c$  commutes with all  $h \in A$ . This proves the lemma.  $\square$

The following corollary constructs a morphism  $\tau_{\underline{g}}$  which maps a given closed subset  $Z$  of some direct power of  $G$  onto a subset of  $G$  of positive dimension.

**Corollary 41.** *Let  $G \leq GL(n, \overline{\mathbb{F}})$  be a linear algebraic group and let  $1 \in \alpha \subset G$  be an ordered finite subset whose centraliser  $\mathcal{C}_G(\alpha)$  is finite. Then for each integer  $m \geq 0$  and each irreducible closed subset  $Z \subset \prod^m G$  of dimension  $\dim(Z) > 0$  there is a sequence  $\underline{g} = (g_1, g_2, \dots, g_m) \in \prod^m \alpha$  such that the closed image  $\overline{\tau_{\underline{g}}(Z)}$  has positive dimension. Moreover, our construction of  $\underline{g}$  is uniquely determined.*

*Proof.* We shall prove the theorem by induction on  $m$ . For  $m = 1$  the statement is obvious. So let  $m \geq 2$  and we assume that the corollary holds whenever the number of factors is smaller than  $m$ . We define several morphisms. For all  $g \in G$  let

$$\sigma_g : \prod^m G \rightarrow \prod^{m-1} G, \quad \sigma_g(a_1, \dots, a_m) = (g^{-1}a_1ga_2, a_3, \dots, a_m)$$

and let

$$\begin{aligned} \pi : \prod^m G &\rightarrow \prod^{m-2} G, & \pi(a_1, \dots, a_m) &= (a_3, a_4, \dots, a_m), \\ \rho : \prod^{m-1} G &\rightarrow \prod^{m-2} G, & \rho(a_2, \dots, a_m) &= (a_3, a_4, \dots, a_m). \end{aligned}$$

For  $m = 2$  we use the convention that  $\prod^0 G$  is a single point. Note, that these morphisms manipulate only the first two coordinates. In particular

$$\rho(\sigma_g(x)) = \pi(x) \quad \text{for all } x \in \prod^m G.$$

Our goal is to find an element  $g \in \alpha$  such that

$$(14) \quad \dim(\overline{\sigma_g(Z)}) > 0.$$

Then we choose the smallest such  $g$  (in the order of  $\alpha$ ) and use the induction hypotheses for  $\overline{\sigma_g(Z)} \subseteq \prod^{m-1} G$ . This proves the corollary for  $Z$  as well.

We distinguish two cases. Suppose first that for all  $z \in \prod^{m-2} G$  the subset  $Z \cap \pi^{-1}(z)$  is finite (i.e. 0 dimensional). Then  $\dim(Z) = \dim(\overline{\pi(Z)})$  is positive (see Fact 20.(e)). But

$$\dim(Z) \geq \dim(\overline{\sigma_g(Z)}) \geq \dim(\overline{\rho(\sigma_g(Z))}) = \dim(\overline{\pi(Z)})$$

hence all these dimensions are equal. Hence (14) is achieved, the corollary holds in this case.

Suppose next that there is a point  $z \in \prod^{m-2}G$  such that  $Z \cap \pi^{-1}(z)$  has an irreducible component  $Z'$  with positive dimension. For simplicity we shall identify the subset  $\pi^{-1}(z) = \prod^2G \times \{z\} \subset \prod^mG$  with  $\prod^2G$  and also  $\rho^{-1}(z) = G \times \{z\} \subset \prod^{m-1}G$  with  $G$ . With these identifications we have

$$\sigma_g(x) = \tau_{(g,1)}(x) \quad \text{for all } x \in \prod^2G \text{ and all } g \in \alpha .$$

If  $\overline{\sigma_1(Z')} = \overline{\tau_{(1,1)}(Z')}$  has positive dimension then (14) holds with  $g = 1$  since  $\dim(\overline{\sigma_1(Z)}) \geq \dim(\overline{\sigma_1(Z')})$ . Otherwise we apply Lemma 39 to our  $Z'$  and get an infinite subset  $A \leq G$ . By assumption  $\alpha$  does not centralise  $A$ , hence there is an element  $g \in \alpha$  which does not commute with  $A$ , i.e.  $g \notin \mathcal{C}_G(A) \cdot 1$ . Now  $\overline{\tau_{(g,1)}(Z')} = \overline{\sigma_g(Z')}$  has positive dimension. But then the potentially larger set  $\overline{\sigma_g(Z)} \supseteq \overline{\sigma_g(Z')}$  has positive dimension as well. In all cases we proved (14), hence the corollary holds.  $\square$

## 6. SPREADING LARGE CONCENTRATION IN A GROUP

In this section we establish our main technical tool, the Spreading Theorem. Roughly speaking it says the following. Let  $\alpha$  be a finite subset in a connected linear algebraic group  $G$  such that  $\mathcal{C}_G(\alpha)$  is finite. If  $G$  has a closed subset  $X$  in which  $\alpha$  has much larger concentration than in  $G$  then we can find a connected closed subgroup  $H \leq G$  normalised by  $\alpha$  in which a small power of  $\alpha$  has similarly large concentration. (When  $G$  is the simple algebraic group used to define a finite group of Lie type  $L$  and  $\alpha$  generates  $L$  then  $H$  turns out to be  $G$  itself.)

**Definition 42.** A finite set  $\alpha \subset GL(n, \overline{\mathbb{F}})$  is called *symmetric* if  $\alpha = \alpha^{-1}$ .

We need the following basic facts.

**Proposition 43.** Let  $\alpha \subset GL(n, \overline{\mathbb{F}})$  be a symmetric subset and  $hH$  a coset of a closed subgroup  $H \leq GL(n, \overline{\mathbb{F}})$ . If  $hH \cap \alpha \neq \emptyset$  then

$$\mu(\alpha^2, hH) \geq \mu(\alpha, H) , \quad \mu(\alpha^2, H) \geq \mu(\alpha, hH) .$$

$\square$

In the rest of this paper we restrict our attention to connected linear algebraic groups. It is not a serious restriction in the light of the following:

**Corollary 44.** *Let  $G \leq GL(n, \overline{\mathbb{F}})$  be a closed subgroup and  $1 \in \alpha \subset GL(n, \overline{\mathbb{F}})$  a finite symmetric subset. Then*

$$\mu(\alpha, G^0) \leq \mu(\alpha, G) \leq \mu(\alpha^2, G^0) + \log(\deg(G)) .$$

*Proof.* It follows from Fact 36.(b) and Proposition 43.  $\square$

**Definition 45.** A *spreading system*  $\alpha|G$  consists of a connected closed subgroup  $G \leq GL(n, \overline{\mathbb{F}})$ , an ordered finite symmetric subset  $1 \in \alpha \subset GL(n, \overline{\mathbb{F}})$  normalising  $G$  such that  $\mu(\alpha, G) \geq 0$  and  $\mathcal{C}_G(\alpha)$  is finite. We say that  $\alpha|G$  is  $(N, \Delta, K)$ -*bounded* for some integer  $N > 0$  and reals  $\Delta > 0, K > 0$  if

$$\dim(G) \leq N, \quad \deg(G) \leq \Delta, \quad \text{mult}(G) \leq \Delta, \quad \text{inv}(G) \leq \Delta, \quad |\alpha \cap G| \geq K.$$

We say that  $\alpha|G$  is  $(\varepsilon, M, \delta)$ -*spreading* for some reals  $\varepsilon > 0, \delta > 0$  and integer  $M > 0$ , if there is a connected closed subgroup  $H \leq G$  normalised by  $\alpha$  such that  $\dim(H) > 0$  and

$$\deg(H) \leq \delta, \quad \mu(\alpha^M, H) \geq (1 + \varepsilon) \cdot \mu(\alpha, G) .$$

Note, that  $\text{mult}(H)$  and  $\text{inv}(H)$  are also bounded in terms of  $\delta$  and  $\Delta$  by Proposition 31. We call such an  $H$  a *subgroup of spreading*, or sometimes *subgroup of  $(\varepsilon, M, \delta)$ -spreading*.

*Remark 46.* Note that the assumption  $\mu(\alpha, G) \geq 0$  is equivalent to  $\dim(G) > 0$  and  $\alpha \cap G \neq \emptyset$ .

Suppose that for some  $m \geq 0$  we find a closed subset  $Z \subseteq \prod^m G$  in which  $\prod^m \alpha$  has large concentration. We use the following lemma to find a closed subset of  $G$  in which the concentration of a small power of  $\alpha$  is almost as large.

**Lemma 47** (Back to  $G$ ). *For all parameters  $N > 0$  and  $\Delta > 0$  there are reals  $B = B_b(N, \Delta) > 0$  and  $K = K_b(N, \Delta) \geq 0$  with the following property.*

*Let  $\alpha|G$  be a spreading system with  $\dim(G) \leq N, \deg(G) \leq \Delta$  and  $\text{mult}(G) \leq \Delta$ . Then for all closed subsets  $Z \subset \prod^m G$  with  $0 < m \leq N, \dim(Z) > 0, \deg(Z) \leq \Delta$  and  $|\prod^m \alpha \cap Z| \geq K$  there is a closed subset  $Y \subseteq G$  such that  $\dim(Y) > 0, \deg(Y) \leq B$  and*

$$\mu(\alpha^{3N}, Y) \geq \mu(\prod^m \alpha, Z) - \log(B) .$$

*Moreover, our construction of  $Y$  is uniquely determined.*

*Proof.* There is nothing to prove for  $m = 1$ , so we assume  $m \geq 2$ . We prove the lemma by induction on  $\dim(Z)$ . This is possible, since  $\dim(Z) \leq N^2$ , so the induction has at most  $N^2$  steps. We assume that the lemma holds in dimensions smaller than  $\dim(Z)$  with some bounds

$B'(N, \Delta, \dim(Z))$  and  $K'(N, \Delta, \dim(Z))$ . By Lemma 25 if  $K$  is large enough then there is a (uniquely determined) positive dimensional irreducible closed set  $Z' \subseteq Z$  of degree  $\deg(Z') \leq B_{\text{irr}}(N^2, \Delta)$  with large concentration:

$$\mu(\prod^m \alpha, Z') \geq \mu(\prod^m \alpha, Z) - \log(B_{\text{irr}}(N^2, \Delta)).$$

This implies immediately that

$$\left| \prod^m \alpha \cap Z' \right| \geq \frac{|\prod^m \alpha \cap Z|^{\dim(Z')/\dim(Z)}}{B_{\text{irr}}(N^2, \Delta)^{\dim(Z')}} \geq \frac{K^{1/N^2}}{B_{\text{irr}}(N^2, \Delta)^{N^2}}.$$

By the above it is enough to complete the induction step for  $Z'$ , so from now on we assume that  $Z$  is irreducible. Corollary 41 gives us a (uniquely determined) sequence  $\underline{g} = (g_1, g_2, \dots, g_m) \in \prod^m \alpha$  such that  $\overline{\tau_{\underline{g}}(Z)}$  has positive dimension. Recall from Remark 34 the bound  $\Delta_{\tau}(N, N, \Delta) \geq \deg(\tau_{\underline{g}})$ . Let

$$\tilde{\Delta} = \max(\Delta, \Delta_{\tau}(N, N, \Delta)).$$

We use Lemma 27 for the two closed sets  $Z \subseteq X = \prod^m G$ , the morphism  $f = \tau_{\underline{g}}$ , the finite set  $\prod^m \alpha$  (denoted by  $\alpha$  in Lemma 27) and  $\varepsilon = 0$ . We note that  $\tau_{\underline{g}}(\prod^m \alpha) \subseteq \alpha^{3N}$ . There are two possible outcomes. In case of Lemma 27.(10) the closed subset  $T = \overline{\tau_{\underline{g}}(Z)} \subseteq G$  satisfies  $\dim(T) > 0$ ,

$$\mu(\prod^m \alpha, Z) - \log(B_{\text{transport}}(\tilde{\Delta})) \leq \mu(\tau_{\underline{g}}(\prod^m \alpha), T) \leq \mu(\alpha^{3N}, T)$$

and by Fact 20.(b) there is an upper bound  $\deg(T) \leq D$  depending only on  $N$  and  $\Delta$ . Hence the lemma holds now with  $Y = T$  and any  $B \geq \max(B_{\text{transport}}(\tilde{\Delta}), D)$ . In case of Lemma 27.(11) we have a closed subset  $S \subseteq Z \subseteq \prod^m G$  with  $0 < \dim(S) < \dim(Z)$ ,  $\deg(S) \leq B_{\text{transport}}(\tilde{\Delta})$  and

$$\mu(\prod^m \alpha, S) \geq \mu(\prod^m \alpha, Z) - \log(B_{\text{transport}}(\tilde{\Delta})).$$

This implies immediately that

$$\left| \prod^m \alpha \cap S \right| \geq \frac{|\prod^m \alpha \cap Z|^{\dim(S)/\dim(Z)}}{B_{\text{transport}}(\tilde{\Delta})^{\dim(S)}} \geq \frac{K^{1/N^2}}{B_{\text{transport}}(\tilde{\Delta})^{N^2}}$$

that is, we can make  $\prod^m \alpha \cap S$  sufficiently large by choosing  $K$  large enough. We set  $B'' = B'(N, B_{\text{transport}}(\tilde{\Delta}), \dim(Z))$  and apply the induction hypothesis to this  $S$ . This gives us a closed set  $Y \subseteq G$  such that  $\dim(Y) > 0$ ,  $\deg(Y) \leq B''$  and

$$\mu(\alpha^{3N}, Y) \geq \mu(\prod^m \alpha, S) - \log(B'') \geq$$

$$\geq \mu(\prod^m \alpha, Z) - \log(B_{\text{transport}}(\tilde{\Delta})B'') ,$$

the lemma holds again with the bound  $B = B_{\text{transport}}(\tilde{\Delta})B''$ . The induction step is complete now, the lemma holds in dimension  $\dim(Z)$ .  $\square$

We are now ready to prove the Spreading Theorem. Let us first give an outline of the proof which avoids technicalities. Suppose that  $\alpha$  has “large” concentration in a subset  $X \subseteq G$ . We would like to “spread” this large concentration as much as possible, i.e. we are looking for a small power  $\alpha^M$  having large concentration in a subgroup  $H$  (more precisely, we need a subgroup of spreading  $H$ ).

We start with  $T_0 = X$  and proceed with a simple induction. Proposition 37 gives us a surjective morphism  $\tau_{\underline{g}}$  which maps  $Z = \prod^{2\dim(G)}(X^{-1} \times X)$  (the direct product of  $2\dim(G)$  copies of the direct product  $(X^{-1} \times X)$ ) onto a subgroup  $H \leq G$ . The concentration of the product set  $\prod^{4\dim(G)} \alpha$  is large in  $Z$ , and we try to transport it via  $\tau_{\underline{g}}$  into  $H$ . Note, that our  $\tau_{\underline{g}}$  maps  $\prod^{4\dim(G)} \alpha$  into a small power  $\alpha^m$ . According to the Transport Lemma 27 we either succeed and therefore  $H$  is a subgroup of spreading, or find a subset  $S \subseteq Z$  with significantly larger concentration. This  $S$  lives in the direct product  $\prod^{4\dim(G)} G$ , but Lemma 47 brings it back to  $G$ , i.e. we find a subset  $T_1 \subseteq G$  such that a small power  $\alpha^{m_1}$  has significantly larger concentration in  $T_1$  than  $\alpha$  had in  $T_0$  (see Lemma 48).

We repeat this process several times. Either at some point we quit the induction with a subgroup of spreading  $H$  or we obtain a sequence of subsets  $T_0, T_1, \dots$  with a quickly growing sequence of concentrations  $\mu(\alpha^{m_i}, T_i)$ . If we let the concentration grow sufficiently large i.e.  $\mu(\alpha^m, T_i) \geq \dim(G)\mu(\alpha, X)$  for some  $i$  then already in  $T_i$  there are enough elements to force large concentration in  $G$ . Therefore we either quit the induction with a subgroup of spreading, or in a bounded number of steps we conclude that  $\mu(\alpha^{m_i}, G)$  is large i.e.  $G$  itself is a subgroup of spreading.

**Lemma 48** (Try to Spread). *For all parameters  $N > 0$  and  $\Delta > 0$  there is an integer  $M_t = M_t(N)$ , and there are reals  $B_t = B_t(N, \Delta) > 0$  and  $K = K_t(N, \Delta) \geq 0$  with the following property. Let  $\alpha|G$  be a spreading system with  $\dim(G) \leq N$ ,  $\deg(G) \leq \Delta$ ,  $\text{mult}(G) \leq \Delta$  and  $\text{inv}(G) \leq \Delta$ . Then for all closed subsets  $Y \subset G$  with  $\dim(Y) > 0$ ,  $\deg(Y) \leq \Delta$  and  $|\alpha \cap Y| \geq K$  and all values*

$$\kappa \geq \log(B_t)$$

at least one of the following holds:

Either there is a connected closed subgroup  $H \leq G$  normalised by  $\alpha$  such that  $\dim(H) > 0$ ,  $\deg(H) \leq B_t$  and

$$(15) \quad \mu(\alpha^{M_t}, H) \geq \mu(\alpha, Y) - \kappa ,$$

or there is a closed set  $T \subseteq G$  such that  $\deg(T) \leq B_t$ ,  $\dim(T) > 0$  and

$$(16) \quad \mu(\alpha^{M_t}, T) \geq \mu(\alpha, Y) + \frac{\kappa}{8N^2} .$$

Moreover, our constructions of  $H$  and  $T$  are uniquely determined.

*Proof.* Using Lemma 25 as in the proof of Lemma 47, we may assume that  $Y$  is irreducible. Let us recall from Lemma 27, Lemma 47, Remark 34 and Proposition 37 the functions  $B_{\text{transport}}$ ,  $B_b$ ,  $\Delta_\tau$  and  $\delta$ . We define the following parameters:

$$\begin{aligned} m &= N \\ \Delta_1 &= \max(\Delta^{6m}, \Delta_\tau(4m, N, \Delta)) \\ B_{\text{transport}} &= B_{\text{transport}}(\Delta_1) \\ \Delta_2 &= \max(\Delta, B_{\text{transport}}) \\ B_b &= B_b(4m, \Delta_2) \\ \varepsilon &= \frac{\kappa}{8mN} + \log(B_{\text{transport}}) + \log(B_b) \\ M_t &= \max(4m^2, 12N) \\ B_t &= \max\left(\delta(N, \Delta), B_{\text{transport}}^{8m(N+1)} \cdot B_b^{8m(N+1)}\right) \end{aligned}$$

We apply Proposition 37. to the subset  $Y$ , this gives us a sequence  $\underline{g} = (g_1, g_2, \dots, g_{2m}) \in \prod^{2m} \alpha^{m-1}$  and a connected closed subgroup  $H \leq G$  normalised by  $\alpha$  such that  $\dim(H) > 0$ ,  $\deg(H) \leq \delta(N, \Delta) \leq B_t$  and

$$\tau_{\underline{g}}(\prod^{2m} Y^{-1} Y) = H .$$

We apply Lemma 27 with parameters  $\Delta_1$  and  $\varepsilon$  to the subsets  $X = \prod^{4m} G$  and  $Z = \prod^{2m} (Y^{-1} \times Y)$ , the morphism  $f = \tau_{(g_1, g_1, g_2, g_2, \dots, g_{2m}, g_{2m})}$ , the finite set  $\prod^{4m} \alpha^{m-1}$  (denoted by  $\alpha$  in Lemma 27). We need to check that all requirements are satisfied. By assumption  $\dim(Y) > 0$  and hence  $\dim(H) = \dim(f(Z)) > 0$ . Since  $Y$  is irreducible,  $Z$  is also irreducible (see Fact 19.(f)) with  $\deg(Z) = \deg(Y)^{4m} \text{inv}(G)^{2m} \leq \Delta^{6m}$  (see Fact 19.(d) and Fact 20.(f)) and  $\deg(f) \leq \Delta_\tau(4m, N, \Delta)$ . Therefore the prerequisites of Lemma 27 are satisfied, hence one of the inequalities 27.(10) or 27.(11) is valid with the logarithmic term equal to  $\log(B_{\text{transport}})$ . Moreover,  $\mu(\prod^{4m} \alpha, Z) = \mu(\alpha, Y)$  and

$$f(\prod^{4m} \alpha) \subseteq \alpha^{4m^2} \subseteq \alpha^{M_t} .$$

In case of 27.(10) we have

$$\begin{aligned}
\mu(\alpha^{M_t}, H) &\geq \mu(\alpha^{4m^2}, f(Z)) \geq \mu(f(\prod^{4m} \alpha), f(Z)) \geq \\
&\geq \mu(\prod^{4m} \alpha, Z) - \log(B_{\text{transport}}) - \varepsilon \cdot \dim(Z) \geq \\
&\geq \mu(\alpha, Y) - \log(B_{\text{transport}}) - \\
&\quad - \left( \frac{\kappa}{8mN} + \log(B_{\text{transport}}) + \log(B_b) \right) \cdot N \cdot 4m \geq \\
&\geq \mu(\alpha, Y) - \frac{\kappa}{2} - 4m(N+1) \left( \log(B_{\text{transport}}) + \log(B_b) \right) \geq \\
&\geq \mu(\alpha, Y) - \frac{\kappa}{2} - \frac{\log(B_t)}{2} \geq \mu(\alpha, Y) - \kappa
\end{aligned}$$

which is exactly inequality (15).

In case of 27.(11) we have a closed subset  $S \subseteq \prod^{4m} G$  with  $\dim(S) > 0$ ,  $\deg(S) \leq B_{\text{transport}}$  such that

$$\begin{aligned}
\mu(\prod^{4m} \alpha, S) &\geq \mu(\prod^{4m} \alpha, Z) - \log(B_{\text{transport}}) + \varepsilon = \\
&= \mu(\alpha, Y) - \log(B_{\text{transport}}) + \left( \frac{\kappa}{8mN} + \log(B_{\text{transport}}) + \log(B_b) \right) = \\
&\geq \mu(\alpha, Y) + \left( \frac{\kappa}{8N^2} + \log(B_b) \right).
\end{aligned}$$

In particular if  $K = K_t(N, \Delta)$  is large enough then

$$|\prod^{4m} \alpha \cap S| \geq |\alpha \cap Y|^{\dim(S)/\dim(Y)} \geq K_b(4m, \Delta_2).$$

We apply Lemma 47 with the parameters  $4N$  and  $\Delta_2$  (which are denoted there by  $N$  and  $\Delta$ ) to the set  $S \subseteq \prod^{4m} G$  (which is denoted there by  $Z$ ). Then in the inequalities we have to use  $B_b = B_b(4m, \Delta_2)$ . Lemma 47 gives us a subset  $T \subseteq G$  (denoted there by  $Y$ ) with  $\dim(T) > 0$ ,  $\deg(T) \leq B_b$  and

$$\begin{aligned}
\mu(\alpha^{12N}, T) &\geq \mu(\prod^{4m} \alpha, S) - \log(B_b) \geq \\
&\geq \mu(\alpha, Y) + \left( \frac{\kappa}{8N^2} + \log(B_b) \right) - \log(B_b) = \mu(\alpha, Y) + \frac{\kappa}{8N^2}
\end{aligned}$$

which implies inequality (16). The lemma is proved in all cases.  $\square$

**Theorem 49** (Spreading Theorem). *For all parameters  $N > 0$ ,  $\Delta > 0$  and  $\frac{1}{3} \geq \varepsilon > 0$  there is an integer  $M = M_{\text{spreading}}(N, \varepsilon)$  and a real  $K = K_{\text{spreading}}(N, \Delta, \varepsilon)$  with the following property.*

*Let  $\alpha|G$  be an  $(N, \Delta, K)$ -bounded spreading system and  $X$  a closed subset in  $\prod^m G$  for some  $0 < m \leq N$ . If  $\deg(X) \leq \Delta$ ,  $\dim(X) > 0$  and*

$$\mu(\prod^m \alpha, X) \geq (1 + 3\varepsilon) \cdot \mu(\alpha, G)$$

*then  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading. Moreover, our construction of the subgroup of spreading is uniquely determined.*

*Proof.* Using Lemma 47 we can easily reduce the theorem to the special case of  $m = 1$ , so we assume  $X \subseteq G$ . Let us recall from Lemma 48 the functions  $M_t$  and  $B_t$ . By induction on  $i \geq 0$  we shall define the following numbers:

$$\Delta_0 = \Delta, \quad \Delta_i = \max(\Delta_{i-1}, B_t(N, \Delta_{i-1})), \quad M_i = M_t(N)^i.$$

Let  $I = I(N, \varepsilon)$  be the smallest positive integer such that

$$(17) \quad \left(1 + \frac{\varepsilon}{4N^2}\right)^I \geq N.$$

We set  $M = M_I$  and

$$K = \max\left(\Delta_I^{N/\varepsilon}, K_t(N, \Delta_0)^N, K_t(N, \Delta_1)^N, \dots, K_t(N, \Delta_{i-1})^N\right).$$

Let  $\alpha|_G$  be an  $(N, \Delta, K)$ -bounded spreading system and  $X \subseteq G$  a closed subset satisfying the conditions of the theorem. Then

$$\mu(\alpha, X) > \mu(\alpha, G) \geq \frac{\log(K)}{N}.$$

By induction on  $i$  we build a series of closed subsets  $T_i \subseteq G$  such that

$$(18) \quad \begin{cases} \dim(T_i) > 0, & \deg(T_i) \leq \Delta_i, \\ \mu(\alpha^{M_i}, T_i) \geq \left(1 + \frac{\varepsilon}{4N^2}\right)^i \cdot \mu(\alpha, X) \geq \frac{\log K}{N}. \end{cases}$$

We run the induction until we either prove Theorem 49 or build the set  $T_I$ . We start the induction with  $T_0 = X$ , this certainly satisfies (18) with  $i = 0$ . In the  $i$ -th step of the induction we assume that  $T_{i-1}$  is already constructed and  $i \leq I$ .

We apply the Lemma 48 with parameters  $N$  and  $\Delta_{i-1}$  to the closed subset  $Y = T_{i-1}$  and to the finite set  $\alpha^{M_{i-1}}$  and

$$\kappa = \varepsilon \cdot \left(1 + \frac{\varepsilon}{4N^2}\right)^{i-1} \cdot \mu(\alpha, X).$$

We need to check that  $\kappa \geq \varepsilon \cdot \mu(\alpha, X) \geq \frac{\varepsilon}{N} \cdot \log(K) \geq \log(\Delta_I) \geq \log(\Delta_i) \geq \log(B_t(N, \Delta_{i-1}))$  and  $|\alpha^{M_{i-1}} \cap T_{i-1}| \geq \exp(\mu(\alpha^{M_{i-1}}, T_{i-1})) \geq K^{1/N} \geq K_t(N, \Delta_{i-1})$ . Note that

$$(\alpha^{M_{i-1}})^{M_t(N)} = \alpha^{M_i} \subseteq \alpha^M.$$

There are two cases. If inequality 48.(16) holds with a subset  $T$  then

$$\begin{aligned} \mu(\alpha^{M_i}, T) &\geq \mu(\alpha^{M_{i-1}}, T_{i-1}) + \frac{\kappa}{4N^2} \geq \\ &\left(1 + \frac{\varepsilon}{4N^2}\right)^{i-1} \cdot \mu(\alpha, X) + \frac{\varepsilon}{4N^2} \left(1 + \frac{\varepsilon}{4N^2}\right)^{i-1} \cdot \mu(\alpha, X) = \\ &= \left(1 + \frac{\varepsilon}{4N^2}\right)^i \cdot \mu(\alpha, X) \end{aligned}$$



and  $\deg(T) \leq B_t(N, \Delta_{i-1}) \leq \Delta_i$  hence  $T_i = T$  satisfies the condition (18). On the other hand, if inequality 48.(15) holds with an appropriate subgroup  $H$  then we find that  $\deg(H) \leq B_t(N, \Delta_{i-1}) \leq \Delta_i \leq K$  and

$$\begin{aligned} \mu(\alpha^M, H) &\geq \mu(\alpha^{M_i}, H) \geq \mu(\alpha^{M_{i-1}}, T_{i-1}) - \kappa \geq \\ &\geq \left(1 + \frac{\varepsilon}{4N^2}\right)^{i-1} \cdot \mu(\alpha, X) - \varepsilon \cdot \left(1 + \frac{\varepsilon}{4N^2}\right)^{i-1} \cdot \mu(\alpha, X) \geq \\ &\geq (1 - \varepsilon) \cdot \mu(\alpha, X) \geq (1 - \varepsilon)(1 + 3\varepsilon)\mu(\alpha, G) \geq (1 + \varepsilon)\mu(\alpha, G). \end{aligned}$$

The theorem holds in this case and we stop the induction.

Finally we consider the case when the induction does not stop during the first  $I$  steps and we build  $T_I$ . Using the first inequality from Proposition 23 and inequalities (18) and (17) we obtain that

$$\begin{aligned} \mu(\alpha^M, G) &\geq \frac{\dim(T_I)}{\dim(G)} \cdot \mu(\alpha^M, T_I) \geq \\ &\geq \frac{1}{N} \cdot \left(1 + \frac{\varepsilon}{4N^2}\right)^I \cdot \mu(\alpha, X) \geq \mu(\alpha, X) \geq (1 + 3\varepsilon)\mu(\alpha, G). \end{aligned}$$

That is,  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading with  $H = G$ . The theorem holds in this case too.  $\square$

## 7. VARIATIONS ON SPREADING

The following useful lemma shows that growth in a subgroup of  $G$  implies growth in  $G$  itself. See [30] for similar results.

**Lemma 50.** *Let  $A \leq G \leq GL(n, \overline{\mathbb{F}})$  be closed subgroups and  $1 \in \alpha \subset GL(n, \overline{\mathbb{F}})$  a finite subset. Then for all integers  $k > 0$  one has*

$$\mu(\alpha^{k+1}, G) \geq \mu(\alpha, G) + \frac{\dim(A)}{\dim(G)} \left[ \mu(\alpha^k, A) - \mu(\alpha^{-1}\alpha, A) \right]$$

or equivalently

$$\frac{|\alpha^{k+1} \cap G|}{|\alpha \cap G|} \geq \frac{|\alpha^k \cap A|}{|\alpha^{-1}\alpha \cap A|}.$$

*Proof.* The two inequalities are clearly equivalent, we shall prove the latter form. We shall look at the multiplication map

$$(\alpha \cap G) \times (\alpha^k \cap A) \xrightarrow{\phi} (\alpha \cap G) \cdot (\alpha^k \cap A) \subseteq (\alpha^{k+1} \cap G)$$

On the left hand side we have  $|\alpha \cap G| \cdot |\alpha^k \cap A|$  elements, on the right hand side there are  $|\alpha^{k+1} \cap G|$  elements. Therefore it is enough to prove that

$$|\phi^{-1}(g)| \leq |\alpha^{-1}\alpha \cap A| \quad \text{for all } g \in \alpha^{k+1} \cap G$$

and this follows from the calculation below:

$$\phi^{-1}(g) \subseteq \{(a, a^{-1}g) \mid a \in \alpha, a^{-1}g \in A\} \subseteq \{(a, a^{-1}g) \mid a \in \alpha \cap gA\},$$

hence

$$|\phi^{-1}(g)| \leq |\alpha \cap gA| \leq |(\alpha \cap gA)^{-1}(\alpha \cap gA)| \leq |\alpha^{-1}\alpha \cap A| .$$

□

The following result is closely related to the “escape from subvarieties” type results in [29] and [30].

**Lemma 51** (Escape Lemma). *For all parameters  $N > 0$ ,  $\Delta > 0$  and  $\frac{1}{7N^2} \geq \varepsilon > 0$  there is an integer  $M = M_{\text{escape}}(N, \varepsilon)$  and a real  $K = K_{\text{escape}}(N, \Delta, \varepsilon)$  with the following property.*

*Let  $\alpha|G$  be an  $(N, \Delta, K)$ -bounded spreading system and  $X \subsetneq Y$  two closed subsets in  $\prod^m G$  for some  $1 \leq m \leq N$ . Suppose that  $\dim(Y) > 0$ ,  $Y$  is irreducible,  $\deg(X) \leq \Delta$  and*

$$\mu(\prod^m \alpha, Y) \geq (1 - \varepsilon) \cdot \mu(\alpha, G) ,$$

$$\mu(\prod^m \alpha, Y \setminus X) \leq (1 - 2\varepsilon) \cdot \mu(\alpha, G) .$$

*Then  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading. Moreover, our construction of the subgroup of spreading is uniquely determined.*

*Proof.* We set  $M = M_{\text{escape}}(N, \varepsilon) = M_{\text{spreading}}(N, \varepsilon)$  and

$$K = K_{\text{escape}}(N, \Delta, \varepsilon) = \max \left( K_{\text{spreading}}(N, \Delta, \varepsilon), 2^{N/\varepsilon}, (2\Delta+1)^{N/(1-\varepsilon)} \right) .$$

Then  $\mu(\alpha, G) \geq \frac{\log(K)}{N} \geq \frac{\log(2)}{\varepsilon}$  and

$$\begin{aligned} \log \left( \frac{|\prod^m \alpha \cap Y|}{|\prod^m \alpha \cap (Y \setminus X)|} \right) &= \dim(Y) \left( \mu(\prod^m \alpha, Y) - \mu(\prod^m \alpha, Y \setminus X) \right) \geq \\ &\geq \dim(Y) \cdot \varepsilon \cdot \mu(\alpha, G) \geq \log(2) . \end{aligned}$$

Therefore  $|\prod^m \alpha \cap X| \geq \frac{1}{2} |\prod^m \alpha \cap Y| \geq \frac{1}{2} |\alpha \cap G|^{(1-\varepsilon)\dim(Y)/\dim(G)} > \Delta$ , hence  $\dim(X) > 0$  and

$$\begin{aligned} \mu(\prod^m \alpha, X) &\geq \frac{\dim(Y)}{\dim(X)} \mu(\prod^m \alpha, Y) - \log(2) \geq \\ &\geq \left( 1 + \frac{1}{\dim(X)} \right) (1-\varepsilon) \cdot \mu(\alpha, G) - \log(2) \geq (1+7\varepsilon)(1-\varepsilon) \cdot \mu(\alpha, G) - \log(2) \geq \\ &\geq (1+5\varepsilon) \cdot \mu(\alpha, G) - \varepsilon \cdot \mu(\alpha, G) > (1+3\varepsilon) \cdot \mu(\alpha, G) . \end{aligned}$$

Then  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading by the Spreading Theorem 49. □

## 8. CENTRALISERS

If  $G$  is a simple algebraic group then a maximal torus  $T$  can be obtained as the connected centraliser of a (regular semisimple) element. Using this it follows that if an appropriate subset  $\alpha \subset G$  does not grow then the concentration of a small power of  $\alpha$  in  $T$  is at least  $\mu(\alpha, G)$ . We first generalise this extremely useful result. Then we define CCC-subgroups and establish some of their basic properties.

Recall from Fact 36 that the degree of any centraliser subgroup is at most  $\deg(G)$ .

**Lemma 52** (Centraliser Lemma). *For all parameters  $N > 0$ ,  $\Delta > 0$  and  $1 \geq \varepsilon > 0$  there is an integer  $M = M_c(N, \varepsilon)$  and a real  $K = K_c(N, \Delta, \varepsilon)$  with the following property. Let  $\alpha|G$  be an  $(N, \Delta, K)$ -bounded spreading system and  $C = \mathcal{C}_G(b_1, b_2, \dots, b_m)$  the centraliser of  $m \leq N$  elements  $b_i \in \alpha \cap G$ . If  $0 < \dim(C)$  then either*

$$\mu(\alpha^M, C^0) \geq (1 - \varepsilon \cdot 8N) \cdot \mu(\alpha, G)$$

*or  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading. Moreover, in the latter case our construction of the subgroup of spreading is uniquely determined.*

*Proof.* We set  $M = M_c(N, \varepsilon) = \max(4, 3M_{\text{spreading}}(N, \varepsilon))$ ,  $\tilde{\Delta} = \max(\Delta, \Delta^{3m})$  and

$$K = K_c(N, \Delta, \varepsilon) = \max\left(\Delta^{1/\varepsilon}, \Delta \cdot K_{\text{spreading}}(N, \tilde{\Delta}, \varepsilon)\right).$$

Note that  $\dim(C^0) = \dim(C) > 0$  and  $|C : C^0| \leq \Delta$  by Fact 36.(b). Combining this with Proposition 43 we obtain that for some  $h \in C$

$$\mu(\alpha^M, C^0) \geq \mu(\alpha^{M/2}, hC^0) \geq \mu(\alpha^{M/2}, C) - \log(\Delta).$$

Since  $K > (\Delta)^{1/\varepsilon}$  we have

$$\mu(\alpha, G) > \frac{1}{\dim(G)} \log(K) \geq \frac{1}{\varepsilon \cdot \dim(G)} \log(\Delta) \geq \frac{1}{\varepsilon \cdot N} \log(\Delta).$$

By the above inequalities it is enough to prove that either  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading or

$$(19) \quad \mu(\alpha^{M/2}, C) \geq (1 - \varepsilon \cdot 7N) \cdot \mu(\alpha, G).$$

If  $\dim(C) = \dim(G)$  then  $G = C$  and there is nothing to prove. So we assume  $\dim(C) < \dim(G)$  and apply Lemma 26 to the subsets  $Z = G$  and  $\alpha$  and to the function

$$f : G \rightarrow \prod^m G, \quad f(g) = (g^{-1}b_1g, g^{-1}b_2g, \dots, g^{-1}b_mg) \in \prod^m G$$

with the parameter  $\varepsilon' = -7\varepsilon \frac{\mu(\alpha, G)}{\dim(C)}$ . The fibres of  $f$  are just the right cosets of the subgroup  $C$ , which have equal dimension, hence we obtain a coset  $S = Ca$  that satisfies inequality (9): either

$$\mu(\alpha, G) \leq \mu(\alpha, Ca) + 7\varepsilon \frac{\mu(\alpha, G)}{\dim(C)} (\dim(G) - \dim(C)) \leq$$

$$\leq \mu(\alpha, Ca) + \varepsilon \cdot 7 \dim(G) \cdot \mu(\alpha, G) \leq \mu(\alpha^2, C) + \varepsilon \cdot 7N \cdot \mu(\alpha, G)$$

(see Proposition 43) and the inequality (19) holds in this case, or else

$$\mu(\alpha, G) \leq \mu(f(\alpha \cap G), \overline{f(G)}) - \frac{7\varepsilon \cdot \mu(\alpha, G)}{\dim(C)} \dim(C) =$$

$$= \mu(f(\alpha \cap G), \overline{f(G)}) - 7\varepsilon \cdot \mu(\alpha, G) .$$

We know  $f(\alpha \cap G) \subseteq \prod^m \alpha^3$  hence in this latter case we have

$$\mu(\prod^m \alpha^3, \overline{f(G)}) \geq (1 + 7\varepsilon) \cdot \mu(\alpha, G) .$$

If  $\mu(\alpha^3, G) \geq (1 + \varepsilon)\mu(\alpha, G)$  then we are done. Otherwise

$$(1 + 3\varepsilon)\mu(\alpha^3, G) \leq (1 + 3\varepsilon)(1 + \varepsilon)\mu(\alpha, G) \leq$$

$$\leq (1 + 7\varepsilon)\mu(\alpha, G) \leq \mu(\prod^m \alpha^3, \overline{f(G)}) .$$

Now  $\deg(\overline{f(G)}) \leq \tilde{\Delta}$  (see Fact 32). We apply the Spreading Theorem 49 with parameters  $N$ ,  $\tilde{\Delta}$  and  $\varepsilon$  to the spreading system  $\alpha^3|G$  and  $X = \overline{f(G)}$ . We obtain that  $\alpha^3|G$  is  $(\varepsilon, \frac{1}{3}M, K)$ -spreading, hence  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading.  $\square$

**Definition 53.** Let  $G$  be an algebraic group and  $X \subseteq G$  an irreducible closed subset. A *CC-generator*<sup>2</sup> for  $X$  is a  $\dim(G)$ -tuple  $\underline{g} \in \prod^{\dim(G)} X$  such that

$$\mathcal{C}_G(\underline{g})^0 = \mathcal{C}_G(X)^0 .$$

Let  $X^{\text{gen}} \subseteq \prod^{\dim(G)} X$  denote the set of all CC-generators and let  $X^{\text{nongen}} = (\prod^{\dim(G)} X) \setminus X^{\text{gen}}$  denote the complement.

Note that  $X^{\text{gen}}$  depends on the group  $G$ , but for simplicity we suppressed it from the notation. When we work with a spreading system  $\alpha|G$  then we always define  $X^{\text{gen}}$  with respect to  $G$ .

**Proposition 54.** *Let  $G$  be an algebraic group and  $X \subseteq G$  an irreducible closed subset. Then  $X$  has a CC-generator i.e.  $X^{\text{gen}} \neq \emptyset$ .*

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<sup>2</sup>CC refers to “connected centraliser”

*Proof.* We consider sequences  $\underline{a} = a_1, a_2, \dots, a_m$ ,  $a_i \in X$  such that

$$G > \mathcal{C}_G(a_1)^0 > \mathcal{C}_G(a_1, a_2)^0 > \mathcal{C}_G(a_1, a_2, a_3)^0 > \dots$$

is a strictly decreasing chain of subgroups. The dimension is strictly decreasing in such a chain, hence the length of  $\underline{a}$  is  $m \leq \dim(G)$ . Therefore one of them, say  $\underline{a}_{\max}$ , is maximal i.e. it cannot be extended. But then

$$\mathcal{C}_G(X)^0 = \mathcal{C}_G(\underline{a}_{\max})^0$$

and we can build a CC-generator from  $\underline{a}_{\max}$  by adding to it  $\dim(G) - m$  arbitrary elements of  $X$ .  $\square$

**Proposition 55.** *Let  $G$  be a connected linear algebraic group,  $X$  an irreducible closed set and  $G \times X \rightarrow X$  a morphism which is a group action. For points  $x \in X$  let  $G_x$  denote the stabiliser subgroup of  $x$ . These are closed subgroups and for each integer  $d$  the subset  $\{x \in X \mid \dim(G_x) > d\}$  is closed in  $X$ . In particular, for each  $d$  the points  $\underline{g} \in \prod^{\dim(G)} G$  with  $\dim(\mathcal{C}_G(\underline{g})) > d$  form a closed subset in  $\prod^{\dim(G)} G$ .*

*Proof.* For the first half of the proposition (about stabiliser subgroups) we refer to [35, Proposition in 1.4]. If we apply this to the conjugation map

$$G \times \prod^{\dim(G)} G \rightarrow \prod^{\dim(G)} G, \quad (h, \underline{g}) \rightarrow h^{-1} \underline{g} h$$

then we obtain the second half (about centraliser subgroups).  $\square$

**Lemma 56.** *Let  $G$  be a connected linear algebraic group and  $\emptyset \neq X \subseteq G$  an irreducible closed subset. Then  $X^{\text{gen}}$  is a dense open subset of  $\prod^{\dim(G)} X$ . Moreover, the degree of its complement  $X^{\text{nongen}}$  is bounded in terms of  $\dim(G)$ ,  $\deg(G)$ ,  $\text{mult}(G)$ ,  $\text{inv}(G)$  and  $\deg(X)$ .*

*Proof.* First of all  $X^{\text{nongen}} = \{\underline{g} \mid \dim(\mathcal{C}_G(\underline{g})) > \dim(A)\}$  is closed by Proposition 55. Its complement  $X^{\text{gen}}$  is naturally open, it is nonempty by Proposition 54, hence it is dense (see Fact 19.(e)).

Let us consider the conjugation map

$$f : G \times \prod^{\dim(G)} X \rightarrow \prod^{\dim(G)} G \times \prod^{\dim(G)} X, \quad f(h, \underline{g}) = (h^{-1} \underline{g} h, \underline{g}).$$

Let  $Y$  denote the diagonal subset

$$Y = \{(\underline{g}, \underline{g}) \mid \underline{g} \in \prod^{\dim(G)} X\} \subset \prod^{\dim(G)} G \times \prod^{\dim(G)} X$$

and let  $\tilde{f}$  denote the restriction of  $f$  to  $f^{-1}(Y)$  composed with the second projection  $Y \rightarrow \prod^{\dim(G)} X$ .

The nonempty fibres of  $f$  can be easily identified with cosets of appropriate centraliser subgroups. Namely, if  $f^{-1}(\underline{g}', \underline{g}) \neq \emptyset$  then  $\underline{g}' = h^{-1}\underline{g}h$  for some element  $h \in G$  and

$$f^{-1}(\underline{g}', \underline{g}) = \mathcal{C}_G(\underline{g})h \times \{\underline{g}\}.$$

All of the involved centralisers contain the subgroup

$$A = \mathcal{C}_G(X)^0$$

and by Proposition 54 at least one of them has dimension  $\dim(A)$ . For  $\underline{g} \in \prod^{\dim(G)} X$  we have  $\underline{g} \in X^{\text{nongen}}$  iff  $\dim(f^{-1}(\underline{g}, \underline{g})) > \dim(A)$ . By Fact 20.(e) the subset

$$Z = \left\{ t \mid \dim(f^{-1}(f(t))) > \dim(A) \right\} \subseteq G \times \prod^{\dim(G)} X$$

is a closed subset and  $\deg(Z)$  is bounded in terms of  $\dim(G)$ ,  $\deg(G)$ ,  $\text{mult}(G)$ ,  $\text{inv}(G)$  and  $\deg(X)$ . By the above  $\tilde{f}(Z \cap f^{-1}(Y)) = X^{\text{nongen}} = \overline{X^{\text{nongen}}}$ . By Fact 20.(f) and Fact 19.(d) we see that  $\deg(X^{\text{nongen}}) = \deg(\overline{f(Z)} \cap Y) \leq \deg(f) \cdot \deg(Z) \cdot \deg(Y)$  which is bounded in terms of  $\dim(G)$ ,  $\deg(G)$ ,  $\text{mult}(G)$ ,  $\text{inv}(G)$  and  $\deg(X)$ .  $\square$

**Definition 57.** Let  $G$  be an algebraic group. A closed subgroup  $A < G$  is a *CCC-subgroup*<sup>3</sup> if  $A = \mathcal{C}_G(X)^0$  for some irreducible closed subset  $X \ni 1$  and  $A$  is different from  $\{1\}$  and  $G^0$ .

**Lemma 58.** Let  $G$  be an algebraic group and  $A < G$  a CCC-subgroup. Then

$$\mathcal{C}_G(\mathcal{C}_G(A)^0)^0 = A, \quad \deg(A) \leq \deg(G)$$

and  $\deg(A^{\text{nongen}})$  is bounded in terms of  $\dim(G)$ ,  $\deg(G)$ ,  $\text{mult}(G)$  and  $\text{inv}(G)$ . If  $B < G$  is another CCC-subgroup with  $A \neq B$  then  $A^{\text{gen}} \cap B^{\text{gen}} = \emptyset$ .

*Proof.* Let  $1 \in X \subseteq G$  be an irreducible closed subset such that  $A = \mathcal{C}_G(X)^0$ . Then  $X \subseteq \mathcal{C}_G(A)^0$ ,  $A$  is connected and commutes with  $\mathcal{C}_G(A)^0$ , hence

$$A = \mathcal{C}_G(X)^0 \supseteq \mathcal{C}_G(\mathcal{C}_G(A)^0)^0 \supseteq A.$$

Now  $\deg(A) \leq \deg(G)$  by Fact 36 and then Lemma 56 implies that  $\deg(A^{\text{nongen}})$  is bounded in terms of  $\dim(G)$ ,  $\deg(G)$ ,  $\text{mult}(G)$  and  $\text{inv}(G)$ . Finally if  $\underline{g} \in A^{\text{gen}}$  then  $\mathcal{C}_G(\mathcal{C}_G(\underline{g})^0)^0 = A \neq B$  hence  $\underline{g} \notin B^{\text{gen}}$ . This proves that  $A^{\text{gen}} \cap B^{\text{gen}} = \emptyset$ .  $\square$

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<sup>3</sup>CCC refers to “connected centraliser of a connected subgroup”

## 9. DICHOTOMY LEMMAS

A central idea of the proof of Theorem 4 for  $L = SL(n, q)$  (as outlined in the introduction) is the following. If a generating set  $\alpha$  of  $L$  does not grow then the intersection of  $\alpha$  with any maximal torus of  $L$  is either relatively large or relatively small. This follows from a similar property of appropriate maximal tori in  $SL(n, \overline{\mathbb{F}}_q)$ . Here we show that CCC-subgroups also satisfy a similar dichotomy. In fact they were designed to do so.

We first prove that if a set  $\alpha$  does not grow (or spread), then for any closed set  $Z$  either the intersection of  $\alpha$  with  $Z$  is relatively small or a small power of  $\alpha$  has relatively large intersection with the centraliser of  $Z$ .

**Lemma 59** (Asymmetric Dichotomy Lemma). *For all parameters  $N > 0$ ,  $\Delta > 0$  and  $\frac{1}{56N^3} > \varepsilon > 0$  there is an integer  $M = M_a(N, \varepsilon)$  and a real  $K = K_a(N, \Delta, \varepsilon)$  with the following property.*

*Let  $\alpha|G$  be an  $(N, \Delta, K)$ -bounded spreading system. Then either  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading or for all irreducible closed subsets  $Z \subseteq G$  such that  $\dim(Z) > 0$ ,  $\deg(Z) < \Delta$  and  $\dim(\mathcal{C}_G(Z)) > 0$  one of the following holds:*

$$\mu(\alpha, Z) < \left(1 - \frac{1}{7N^2}\right) \cdot \mu(\alpha, G)$$

or

$$\mu(\alpha^M, \mathcal{C}_G(Z)^0) \geq \mu\left(\prod^{\dim(G)} \alpha^M, (\mathcal{C}_G(Z)^0)^{\text{gen}}\right) \geq (1 - \varepsilon \cdot 16N) \cdot \mu(\alpha, G).$$

*Moreover, our construction of the subgroup of spreading is uniquely determined.*

*Proof.* We define the parameters

$$\varepsilon' = \frac{1}{7N^2}, \quad \varepsilon''' = \varepsilon \cdot 8N \leq \frac{1}{7N^2}$$

and the closed subsets

$$\begin{aligned} Y' = \prod^{\dim(G)} Z &\supseteq X' = Z^{\text{nongen}} \\ Y''' = \prod^{\dim(G)} \mathcal{C}_G(Z)^0 &\supseteq X''' = (\mathcal{C}_G(Z)^0)^{\text{nongen}}. \end{aligned}$$

We know from Fact 36 that  $\deg(\mathcal{C}_G(Z)^0) \leq \Delta$ . By Lemma 58 there is an upper bound  $\tilde{\Delta} \geq \Delta$  for  $\deg(X')$  and  $\deg(X''')$  which depends only on  $N$  and  $\Delta$ . We set  $M'' = M_c(N, \varepsilon)$ ,

$$M = M_a(N, \varepsilon) = \max\left(M_{\text{escape}}(N, \varepsilon'), M'', M'' \cdot M_{\text{escape}}(N, \varepsilon''')\right)$$

and

$$K = K_a(N, \Delta, \varepsilon) =$$

$$= \max \left( K_{\text{escape}}(N, \tilde{\Delta}, \varepsilon'), K_c(N, \Delta, \varepsilon), K_{\text{escape}}(N, \tilde{\Delta}, \varepsilon''') \right).$$

We apply the Escape Lemma 51 with the parameters  $N$ ,  $\tilde{\Delta}$  and  $\varepsilon'$  to the subsets  $X'$  and  $Y'$ . If the Escape Lemma 51 gives us a subgroup of  $(\varepsilon', M_{\text{escape}}(N, \varepsilon'), K_{\text{escape}}(N, \tilde{\Delta}, \varepsilon'))$ -spreading then the lemma holds since  $\varepsilon \leq \varepsilon'$ . Otherwise there are two possibilities. Either

$$\mu(\alpha, Z) = \mu \left( \prod^{\dim(G)} \alpha, Y' \right) < (1 - \varepsilon') \cdot \mu(\alpha, G) = \left( 1 - \frac{1}{7N^2} \right) \cdot \mu(\alpha, G)$$

in which case the lemma holds, or else there is at least one  $\dim(G)$ -tuple  $\underline{g} \in \prod^{\dim(G)} \alpha \cap Z^{\text{gen}}$  (in fact the Escape Lemma gives us many such tuples). We select the lexicographically minimal  $\underline{g}$  among them. Note that  $\mathcal{C}_G(\underline{g})^0 = \mathcal{C}_G(Z)^0 \neq \{1\}$ , in particular  $\dim(\mathcal{C}_G(\underline{g})) > 0$ . In this latter case we apply the Centraliser Lemma 52 with parameters  $N$ ,  $\Delta$  and  $\varepsilon$  to the spreading system  $\alpha|G$  and the subgroup  $C = \mathcal{C}_G(\underline{g})$ . In case we obtain a subgroup of spreading, the lemma holds. Otherwise we have

$$\mu(\alpha^{M''}, \mathcal{C}_G(Z)^0) \geq (1 - \varepsilon \cdot 8N) \cdot \mu(\alpha, G) = (1 - \varepsilon''') \cdot \mu(\alpha, G).$$

Finally we apply the Escape Lemma 51 with parameters  $N$ ,  $\tilde{\Delta}$  and  $\varepsilon'''$  to the spreading system  $\alpha^{M''}|G$  and the subsets  $X'''$  and  $Y'''$ . Again, the lemma holds if we obtain a subgroup of spreading. Otherwise we have

$$\mu \left( \prod^{\dim(G)} \alpha^{M''}, (\mathcal{C}_G(Z)^0)^{\text{gen}} \right) > (1 - 2\varepsilon''') \cdot \mu(\alpha, G) = (1 - \varepsilon \cdot 16N) \mu(\alpha, G).$$

Then the lemma follows from Proposition 23 via the following calculation:

$$\begin{aligned} \mu(\alpha^M, \mathcal{C}_G(Z)^0) &= \mu \left( \prod^{\dim(G)} \alpha^M, Y''' \right) \geq \\ &\geq \mu \left( \prod^{\dim(G)} \alpha^M, (Y''' \setminus X''') \right) = \mu \left( \prod^{\dim(G)} \alpha^M, (\mathcal{C}_G(Z)^0)^{\text{gen}} \right). \end{aligned}$$

□

The connected centraliser of the connected centraliser of a CCC-subgroup  $A$  is  $A$  itself, hence applying the previous lemma twice we obtain the following.

**Lemma 60** (Dichotomy Lemma). *For all parameters  $N > 0$ ,  $\Delta > 0$  and  $\frac{1}{112N^3} > \varepsilon > 0$  there is an integer  $M = M_{\text{dichotomy}}(N, \varepsilon)$  and a real  $K = K_{\text{dichotomy}}(N, \Delta, \varepsilon)$  with the following property.*

*Let  $\alpha|G$  be an  $(N, \Delta, K)$ -bounded spreading system. Then either  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading or for all CCC-subgroups  $A < G$  one of the following holds:*

$$\mu(\alpha, A) < \left( 1 - \frac{1}{7N^2} \right) \cdot \mu(\alpha, G)$$



or else

$$\mu(\alpha^M, A) \geq \mu\left(\prod^{\dim(G)} \alpha^M, A^{\text{gen}}\right) \geq (1 - \varepsilon \cdot 16N) \cdot \mu(\alpha, G).$$

Moreover, our construction of the subgroup of spreading is uniquely determined.

*Proof.* We set  $M' = M_a(N, \varepsilon)$ ,  $M = M_{\text{dichotomy}}(N, \varepsilon) = (M')^2$  and

$$K = K_{\text{dichotomy}}(N, \Delta, \varepsilon) = K_a(N, \Delta, \varepsilon).$$

We apply the Asymmetric Dichotomy Lemma 59 with parameters  $N$ ,  $\Delta$  and  $\varepsilon$  to  $\alpha|G$  and the irreducible subset  $Z' = A$ . Note that  $\dim(A) > 0$  and  $\dim(\mathcal{C}_G(A)) > 0$  follows from Definition 57. If we obtain a subgroup of  $(\varepsilon, M', K)$ -spreading or if

$$\mu(\alpha, A) < \left(1 - \frac{1}{7N^2}\right) \cdot \mu(\alpha, G)$$

then the lemma holds. Otherwise we have

$$\mu(\alpha^{M'}, \mathcal{C}_G(A)^0) \geq (1 - \varepsilon \cdot 16N) \cdot \mu(\alpha, G).$$

We apply again the Asymmetric Dichotomy Lemma 59 with parameters  $N$ ,  $\Delta$  and  $\varepsilon$  to  $\alpha^{M'}|G$  and  $Z'' = \mathcal{C}_G(A)^0$ . If we obtain a subgroup of  $(\varepsilon, M', K)$ -spreading then it is a subgroup of  $(\varepsilon, M, K)$ -spreading for  $\alpha|G$  and the lemma holds. Otherwise  $\alpha^{M'}|G$  and  $Z''$  must satisfy one of the two inequalities of that lemma. The first one is

$$\mu(\alpha^{M'}, \mathcal{C}_G(A)^0) < \left(1 - \frac{1}{7N^2}\right) \cdot \mu(\alpha, G) \leq (1 - \varepsilon \cdot 16N) \cdot \mu(\alpha, G),$$

but this has already been ruled out. Therefore the other inequality holds:

$$\begin{aligned} & \mu\left((\alpha^{M'})^{M'}, \mathcal{C}_G(\mathcal{C}_G(A)^0)^0\right) \geq \\ & \geq \mu\left(\prod^{\dim(G)} \alpha^{M' \cdot M'}, \left(\mathcal{C}_G(\mathcal{C}_G(A)^0)^0\right)^{\text{gen}}\right) \geq (1 - \varepsilon \cdot 16N) \mu(\alpha, G). \end{aligned}$$

But  $\mathcal{C}_G(\mathcal{C}_G(A)^0)^0 = A$  and the Dichotomy Lemma 60 follows.  $\square$

## 10. FINDING AND USING CCC-SUBGROUPS

Let  $G$  be a simple algebraic group and  $T$  a maximal torus of  $G$ . Combining the previously developed techniques we can show that if an appropriate finite subset  $\alpha \subset G$  does not grow then either  $\mu(\alpha, T)$  is relatively small or  $\alpha$  itself must be very large compared to  $\langle \alpha \rangle$  (which must be finite in this case). We actually prove a similar result for non-normal CCC-subgroups of arbitrary connected linear algebraic groups  $G$ . For  $G$  non-nilpotent we then construct CCC-subgroups which can be used as an input for the above result.

It is crucial in the proofs of our main theorems to find sufficiently many  $\langle \alpha \rangle$ -conjugates of a CCC-subgroup  $A \leq G$ . We define a quantity  $\hat{\mu}$  measuring their number in a sense analogous to the concentration  $\mu$ . To simplify the notation we restrict this definition to the case  $\alpha \subset G$ , in the more general situation we use a much cruder estimate.

**Definition 61.** Let  $G$  be a connected linear algebraic group,  $A \leq G$  a closed subgroup and  $\alpha \subset G$  a finite subset. Suppose that  $G$  does not normalise  $A$ . We define

$$\hat{\mu}(\langle \alpha \rangle, G, A) = \frac{\log \left| \{t^{-1}At \mid t \in \langle \alpha \rangle\} \right|}{\dim(G) - \dim(\mathcal{N}_G(A))} = \frac{\log \left| \langle \alpha \rangle : \mathcal{N}_{\langle \alpha \rangle}(A) \right|}{\dim(G) - \dim(\mathcal{N}_G(A))}.$$

*Remark 62.* The  $G$ -conjugates of  $A$  are parametrised by the quotient variety  $X = G/\mathcal{N}_G(A)$ . Let  $\hat{\alpha} \subset X$  denote the image of  $\langle \alpha \rangle$ , these are the parameter values that correspond to the  $\langle \alpha \rangle$ -conjugates of  $A$ . Then  $\hat{\mu}(\langle \alpha \rangle, G, A) = \mu(\hat{\alpha}, X)$ .

**Lemma 63** (spreading via CCC-subgroups). *For all parameters  $N > 0$ ,  $\Delta > 0$  and  $\frac{1}{119N^3} > \varepsilon > 0$  there is an integer  $M = M_s(N, \varepsilon)$  and a real  $K = K_s(N, \Delta, \varepsilon)$  with the following property.*

*Let  $\alpha|G$  be an  $(N, \Delta, K)$ -bounded spreading system and  $A < G$  a CCC-subgroup such that*

$$\mu(\alpha, A) > \left(1 - \frac{1}{7N^2}\right) \cdot \mu(\alpha, G).$$

*Suppose that at least one of the following holds:*

(a)

$$\left| \langle \alpha \rangle : \mathcal{N}_{\langle \alpha \rangle}(A) \right| \geq |\alpha|^{2N},$$

(b)  $\alpha \subset G$ ,  $A$  is not normal in  $G$  and

$$\mu(\alpha, G) \leq \left(1 - \varepsilon \cdot 64N^3\right) \cdot \hat{\mu}(\langle \alpha \rangle, G, A).$$

*Then  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading. Moreover, our construction of the subgroup of spreading is uniquely determined.*

*Proof.* By Lemma 58 the conjugate subsets  $h^{-1}A^{\text{gen}}h$  for various  $h$  normalising  $G$  are pairwise disjoint or coincide. They are all contained in  $\prod^{\dim(G)} G$  which has dimension  $\dim(G)^2 \leq N^2$ .

In case (b) we consider the following set:

$$X = \bigcup \left\{ h^{-1}A^{\text{gen}}h \mid h \in G \right\} \subseteq \prod^{\dim(G)} G.$$

Then  $\dim(\overline{X}) \leq N^2$ . The virtue of this estimate is that it depends only on  $N$ , but we also need a precise calculation in terms of  $A$  and

$G$ . We consider the conjugation map  $\phi : G \times \overline{A^{\text{gen}}} \rightarrow \prod^{\dim(G)} G$  defined as  $\phi(h, \underline{a}) = h^{-1} \underline{a} h$  (note that  $\overline{A^{\text{gen}}} = \prod^{\dim(G)} A$ ). By definition  $X = \phi(G \times A^{\text{gen}})$  hence  $\overline{X} = \overline{\text{im}(\phi)}$  and  $\deg(\overline{X})$  is bounded in terms of  $N$  and  $\Delta$  (see Fact 20.(f)). Consider any pair  $(h_0, \underline{a}_0) \in G \times A^{\text{gen}}$  and its image  $x = h_0^{-1} \underline{a}_0 h_0 \in X$ . The corresponding fibre is

$$\phi^{-1}(x) = \left\{ (nh_0, n\underline{a}_0 n^{-1}) \mid n \in \mathcal{N}_G(A) \right\},$$

which is isomorphic (as an algebraic set, see Remark 18) to  $\mathcal{N}_G(A)$ . In particular,  $G \times A^{\text{gen}}$  (which is open and dense in the domain of  $\phi$ ) is the union of fibres of dimension  $\dim(\mathcal{N}_G(A))$ . Therefore

$$\dim(\overline{X}) = \dim(A^{\text{gen}}) + \left[ \dim(G) - \dim(\mathcal{N}_G(A)) \right] > \dim(A^{\text{gen}})$$

(apply Fact 20.(e) to the irreducible set  $G \times \overline{A^{\text{gen}}}$ ).

In case (a) we define the parameters  $\varepsilon'' = \varepsilon \cdot 16N > \varepsilon$  and  $\Delta'' = \Delta^N$ , in case (b) we use the same  $\varepsilon''$  and we set  $\Delta'' = \max(\Delta, \deg(\overline{X}))$ . We define

$$M' = M_{\text{dichotomy}}(N, \varepsilon), \quad M'' = M_{\text{spreading}}(N, \varepsilon''),$$

$$M = \max(4M' + 1, 2M' \cdot M''),$$

$$K = \max(K_{\text{dichotomy}}(N, \Delta, \varepsilon), K_{\text{spreading}}(N, \Delta'', \varepsilon'')).$$

We consider all the conjugate subgroups

$$\mathcal{A} = \left\{ t^{-1} A t \mid t \in \langle \alpha \rangle \right\},$$

they are all CCC-subgroups of  $G$  since  $\alpha$  normalises  $G$ .

In case (a) we have  $\log |\mathcal{A}| \geq 2N \log |\alpha|$  by assumption. In case (b) we obtain instead the following estimate

$$\begin{aligned} \log |\mathcal{A}| &= \hat{\mu}(\langle \alpha \rangle, G, A) \cdot \left[ \dim(G) - \dim(\mathcal{N}_G(A)) \right] = \\ &= \left[ \dim(\overline{X}) - \dim(A^{\text{gen}}) \right] \cdot \hat{\mu}(\langle \alpha \rangle, G, A) \geq \\ &\geq \left[ \dim(\overline{X}) - \dim(A^{\text{gen}}) \right] \cdot \frac{1}{1 - \varepsilon \cdot 64N^3} \cdot \mu(\alpha, G) > \\ &> \left[ \dim(\overline{X}) - \dim(A^{\text{gen}}) \right] \cdot (1 + \varepsilon'' \cdot 4 \dim(\overline{X})) \cdot \mu(\alpha, G). \end{aligned}$$

Suppose first that

$$(20) \quad \mu(\alpha^2, B) \geq \left(1 - \frac{1}{7N^2}\right) \mu(\alpha, G)$$

for all  $B \in \mathcal{A}$ . We apply the Dichotomy Lemma 60 with parameters  $N$ ,  $\Delta$  and  $\varepsilon$  to  $\alpha^2|G$  and each  $B \in \mathcal{A}$ . We get that either  $\alpha^2|G$  is

$(\varepsilon, M', K)$ -spreading i.e.  $\alpha|G$  is  $(\varepsilon, 2M', K)$ -spreading, and the lemma holds, or

$$\mu\left(\prod^{\dim(G)} \alpha^{2M'}, B^{\text{gen}}\right) \geq (1 - \varepsilon \cdot 16N) \mu(\alpha^2, G) \geq (1 - \varepsilon'') \mu(\alpha, G)$$

for all  $B \in \mathcal{A}$  (in particular,  $\prod^{\dim(G)} \alpha^{2M'}$  has at least one element in each  $B^{\text{gen}}$ ). Let us consider this latter possibility. By Lemma 58 the subsets  $B^{\text{gen}}$  are pairwise disjoint. In case (b) we obtain

$$\begin{aligned} \mu\left(\prod^{\dim(G)} \alpha^{2M'}, \overline{X}\right) &= \frac{1}{\dim(\overline{X})} \log \left| \prod^{\dim(G)} \alpha^{2M'} \cap \overline{X} \right| \geq \\ &\geq \frac{1}{\dim(\overline{X})} \log \left( \sum_{B \in \mathcal{A}} \left| \prod^{\dim(G)} \alpha^{2M'} \cap B^{\text{gen}} \right| \right) \geq \\ &\geq \frac{1}{\dim(\overline{X})} \left[ \log |\mathcal{A}| + \log \left( \min_{B \in \mathcal{A}} \left| \prod^{\dim(G)} \alpha^{2M'} \cap B^{\text{gen}} \right| \right) \right] = \\ &\geq \frac{1}{\dim(\overline{X})} \left[ \log |\mathcal{A}| + \dim(A^{\text{gen}}) \cdot \min_{B \in \mathcal{A}} \left( \mu\left(\prod^{\dim(G)} \alpha^{2M'}, B^{\text{gen}}\right) \right) \right] \geq \\ &\geq \frac{1}{\dim(\overline{X})} \left[ \log |\mathcal{A}| + \dim(A^{\text{gen}}) \cdot (1 - \varepsilon'') \mu(\alpha, G) \right] \geq \\ &\geq \frac{1}{\dim(\overline{X})} \left[ [\dim(\overline{X}) - \dim(A^{\text{gen}})] \cdot (1 + \varepsilon'' \cdot 4 \dim(\overline{X})) \mu(\alpha, G) + \right. \\ &\quad \left. + \dim(A^{\text{gen}}) \cdot (1 - \varepsilon'') \mu(\alpha, G) \right] = \\ &= \left[ 1 + 4\varepsilon'' (\dim(\overline{X}) - \dim(A^{\text{gen}})) - \varepsilon'' \frac{\dim(A^{\text{gen}})}{\dim(\overline{X})} \right] \mu(\alpha, G) > \\ &> (1 + 3\varepsilon'') \cdot \mu(\alpha, G). \end{aligned}$$

In case (a) a similar, but much shorter calculation shows that

$$\mu\left(\prod^{\dim(G)} \alpha^{2M'}, \prod^{\dim(G)} G\right) \geq \frac{\log |\mathcal{A}|}{\dim(G)^2} \geq \frac{2 \log |\alpha|}{\dim(G)} \geq (1 + 3\varepsilon'') \mu(\alpha, G).$$

In both cases we apply the Spreading Theorem 49 with parameters  $N$ ,  $\Delta''$  and  $\varepsilon''$  to  $\alpha^{2M'}|G$ , and in case (a) to the set  $\prod^{\dim(G)} G$ , in case (b) to the set  $\overline{X}$ . We obtain that  $\alpha^{2M'}|G$  is  $(\varepsilon'', M'', K)$ -spreading, hence  $\alpha|G$  is  $(\varepsilon, 2M'M'', K)$ -spreading, the lemma holds.

Finally we assume that condition (20) does not hold for all members of  $\mathcal{A}$ . As the subgroup  $A$  itself satisfies it, there must be at least one subgroup  $B_0 \in \mathcal{A}$  and an element  $b \in \alpha$  such that  $B_0$  satisfies (20) but  $b^{-1}B_0b$  doesn't:

$$(21) \quad \mu(\alpha^2, b^{-1}B_0b) < \left(1 - \frac{1}{7N^2}\right) \mu(\alpha, G).$$

Conjugating by  $b$  we transform (20) into

$$\mu(\alpha^4, b^{-1}B_0b) \geq \mu(b^{-1}\alpha^2b, b^{-1}B_0b) =$$

$$= \mu(\alpha^2, B_0) > \left(1 - \frac{1}{7N^2}\right) \mu(\alpha, G) .$$

Again we apply the Dichotomy Lemma 60 with parameters  $N$ ,  $\Delta$  and  $\varepsilon$  to  $\alpha^4|G$  and the CCC-subgroup  $b^{-1}B_0b$ . We obtain that either  $\alpha^4|G$  is  $(\varepsilon, M', K)$ -spreading, and the lemma holds in this case, or

$$\mu(\alpha^{4M'}, b^{-1}B_0b) \geq (1 - \varepsilon \cdot 16N) \mu(\alpha, G) .$$

Now we compare this to inequality (21) and apply Lemma 50 to the subgroup  $b^{-1}B_0b$  with  $k = 4M'$ . We obtain that

$$\begin{aligned} \mu(\alpha^{4M'+1}, G) &\geq \mu(\alpha, G) + \frac{\dim(b^{-1}B_0b)}{\dim(G)} \left[ \frac{1}{7N^2} - \varepsilon \cdot 16N \right] \mu(\alpha, G) \geq \\ &\geq \mu(\alpha, G) + \frac{1}{N} \left[ \frac{1}{7N^2} - \varepsilon \cdot 16N \right] \mu(\alpha, G) = \\ &= \left[ 1 + \frac{1}{7N^3} - 16\varepsilon \right] \mu(\alpha, G) \geq (1 + \varepsilon) \mu(\alpha, G) , \end{aligned}$$

hence  $G$  itself is a subgroup of  $(\varepsilon, 4M' + 1, K)$ -spreading for  $\alpha|G$ .  $\square$

**Lemma 64.** *Let  $G$  be a non-abelian connected linear algebraic group and  $\mathcal{S} \subseteq G$  the closure of the set of those elements  $g \in G$  whose centraliser is either the whole of  $G$  or does not contain any maximal torus. Then  $\dim(\mathcal{S}) < \dim(G)$  and the degree of  $\mathcal{S}$  is bounded:*

$$\deg(\mathcal{S}) \leq \Delta_{\text{bad}}(\dim(G), \deg(G)) .$$

*Proof.* Let  $A \leq G$  be a Cartan subgroup. Then  $A = \mathcal{C}_G(T)$  for some maximal torus  $T \leq G$ . Hence for each  $g \in A$  we have  $T \leq \mathcal{C}_G(g)$ . All Cartan subgroups are conjugates of  $A$ , hence their union, denoted by  $\mathcal{R}$ , is the image of the conjugation map  $f : A \times G \rightarrow G$ ,  $f(a, g) = g^{-1}ag$ . It is well-known that  $\mathcal{R}$  contains an open subset  $U$  of  $G$  and by definition  $\overline{G \setminus \mathcal{R}} \subseteq G \setminus U$ , so  $\dim(\overline{G \setminus \mathcal{R}}) < \dim(G)$  (see Fact 19.(e)). Moreover,  $\deg(\overline{G \setminus \mathcal{R}})$  is bounded in terms of  $\dim(G)$  and  $\deg(G)$  (see Fact 20.(d)). We also know that  $\deg(\mathcal{Z}(G)) \leq \deg(G)$  (see Fact 36). Hence  $\mathcal{S} = (\overline{G \setminus \mathcal{R}}) \cup \mathcal{Z}(G)$  also has bounded degree.  $\square$

**Lemma 65** (Finding CCC-subgroups). *For all parameters  $N > 0$ ,  $\Delta > 0$  and  $\frac{1}{56N^3} > \varepsilon > 0$  there is an integer  $M = M_{\text{CCC}}(N, \varepsilon)$  and a real  $K = K_{\text{CCC}}(N, \Delta, \varepsilon)$  with the following property.*

*Let  $\alpha|G$  be an  $(N, \Delta, K)$ -bounded spreading system such that  $G$  is non-nilpotent. Then either it is  $(\varepsilon, M, K)$ -spreading, or there is a CCC-subgroup  $A \leq G$  which contains exactly one maximal torus of  $G$  and satisfies*

$$\mu(\alpha^M, A) > (1 - \varepsilon \cdot 16N) \mu(\alpha, G) .$$

*In particular,  $A$  is not normal in  $G$ . Moreover, our construction of  $A$  and of the subgroup of spreading is uniquely determined.*

*Proof.* Recall the functions  $M_{\text{escape}}, K_{\text{escape}}, M_c, K_c, M_a, K_a$  and  $\Delta_{\text{bad}}$  from the lemmas 51, 52, 59 and 64. We define the following constants:

$$M_c = M_c(N, \varepsilon), \quad M_{\text{escape}} = M_{\text{escape}}(N, \varepsilon), \quad M_a = M_a(N, \varepsilon),$$

$$\tilde{\Delta} = \max(\Delta, \Delta_{\text{bad}}(N, \Delta)), \quad M = M_c^N \max(M_{\text{escape}}, M_a),$$

$$K = \max(K_c(N, \Delta, \varepsilon), K_{\text{escape}}(N, \tilde{\Delta}, \varepsilon), K_a(N, \Delta, \varepsilon)).$$

Set  $g_0 = 1 \in G$ ,  $G_0 = G$ . We define by induction on  $i$  the elements  $g_i \in \alpha^{(M_c)^{i-1}} \cap G$  in such a way that the subgroups

$$G_i = \mathcal{C}_G(g_0, g_1, g_2, \dots, g_i)^0 = \mathcal{C}_{G_{i-1}}(g_i)^0$$

satisfy

$$(22) \quad \mu(\alpha^{(M_c)^i}, G_i) \geq (1 - \varepsilon \cdot 8N) \mu(\alpha, G),$$

all  $G_i$  contain some maximal torus of  $G$  and they form a strictly decreasing series of subgroups. Then their dimension is strictly decreasing as well, hence the sequence has length smaller than  $N$ .

Suppose that such a  $G_i$  is already defined for some  $N > i \geq 0$ . If it is abelian then we stop the induction, otherwise continue. Let  $\mathcal{S}_i \subsetneq G_i$  be the subset defined in Lemma 64. Note, that  $\deg(G_i) \leq \Delta$ ,  $\text{mult}(G_i) \leq \Delta$  and  $\text{inv}(G_i) \leq \Delta$  (see Fact 36), hence  $\deg(\mathcal{S}_i) \leq \tilde{\Delta}$ . We apply the Escape Lemma 51 with parameters  $N, \tilde{\Delta}$  and  $\varepsilon$  to  $\alpha^{(M_c)^i}|G$  and the subsets  $X = \mathcal{S}_i$  and  $Y = G_i$  of  $G$ . If we obtain a subgroup of  $(\varepsilon, M_{\text{escape}}, K)$ -spreading then the lemma holds. Otherwise, since (22) holds, we find at least one element

$$g_{i+1} \in \alpha^{(M_c)^i} \cap (G_i \setminus \mathcal{S}_i).$$

(In fact the Escape Lemma gives us many such elements). We select the  $g_{i+1}$  which is minimal in the order of  $\langle \alpha \rangle$ . According to the definition of  $\mathcal{S}_i$ ,

$$G_{i+1} = (G_i \cap \mathcal{C}_G(g_{i+1}))^0$$

contains a maximal torus of  $G_i$ , which is also a maximal torus in  $G$ , and  $G_{i+1}$  is strictly smaller than  $G_i$ . We apply the Centraliser Lemma 52 with parameters  $N, \Delta$  and  $\varepsilon$  to  $\alpha^{(M_c)^i}|G$  and the centraliser subgroup  $\mathcal{C}_G(g_0, \dots, g_{i+1})$ . In case we obtain a subgroup of  $(\varepsilon, M_c, K)$ -spreading, the lemma holds. Otherwise we have

$$\mu(\alpha^{(M_c)^i M_c}, G_{i+1}) \geq (1 - \varepsilon \cdot 8N) \cdot \mu(\alpha^{(M_c)^i}, G) \geq (1 - \varepsilon \cdot 8N) \mu(\alpha, G)$$

i.e.  $G_{i+1}$  satisfies (22).

As we explained before, this process must stop in at most  $N$  steps. But the only way it can stop is to arrive at a connected abelian subgroup  $G_I$  which contains a maximal torus  $T$  and satisfies inequality (22).

We set  $A = \mathcal{C}_G(G_I)^0$ . On the one hand,  $T$  commutes with  $G_I$ , hence  $T \leq A$ . On the other hand,  $A = \mathcal{C}_G(G_I)^0 \leq \mathcal{C}_G(T)$ , and the latter one is a Cartan subgroup, which has a unique maximal torus. Therefore  $T$  is the only maximal torus in  $A$ . But  $G$  is non-nilpotent, hence  $G$  has several maximal tori. This implies that  $A$  is a CCC-subgroup which is not normal. We apply the Asymmetric Dichotomy Lemma 59 with parameters  $N$ ,  $\Delta$  and  $\varepsilon$  to  $\alpha^{(M_c)^N}|G$  and the subset  $Z = G_I$ . In case we obtain a subgroup of  $(\varepsilon, M_a, K)$ -spreading, the lemma holds. Otherwise, since  $G_I$  satisfies (22), we obtain that

$$\mu(\alpha^M, A) \geq (1 - \varepsilon \cdot 16N) \mu(\alpha, G)$$

as required.  $\square$

Suppose we want to prove that a certain spreading system  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading. Our strategy is to obtain a CCC-subgroup  $A < G$  via Lemma 65, and use Lemma 63 to establish the  $(\varepsilon, M, K)$ -spreading. In order to do this, we need to estimate the number of  $\langle \alpha \rangle$ -conjugates of  $A$ . In Section 11 we develop a powerful method for finite  $\langle \alpha \rangle$ . Later in Section 13 we deal with the much simpler case when  $A$  has infinitely many conjugates.

## 11. FINITE GROUPS OF LIE TYPE

In this section we use the general results established earlier to prove Theorem 6, our main technical result concerning fixpoint groups of Frobenius maps of linear algebraic groups.

**Definition 66.** Let  $G$  be a linear algebraic group over the field  $\overline{\mathbb{F}_p}$ .

- (a) For each  $p$ -power  $q$  the usual  $q$ -th power map  $\overline{\mathbb{F}_p} \rightarrow \overline{\mathbb{F}_p}$  is a field automorphism. Applying this to the entries of the  $n \times n$  matrices we obtain the group automorphisms

$$Frob_q : GL(n, \overline{\mathbb{F}_p}) \rightarrow GL(n, \overline{\mathbb{F}_p}) .$$

(Note, that these are not morphisms of varieties.)

- (b) More generally,  $Frob_q$  can be defined the same way on any algebraically closed field of characteristic  $p$ , hence we can talk about  $Frob_q$ -invariant algebraic sets and  $Frob_q$ -equivariant morphisms (i.e. morphisms compatible with the  $Frob_q$ -actions on the domain and the range). (These are precisely the algebraic sets and morphisms *defined over*  $\mathbb{F}_q$ .)

- (c) A *Frobenius map* of  $G$  is a group automorphism  $\sigma : G \rightarrow G$  such that there is a  $p$ -power  $q$ , an exponent  $k$  and a faithful representation  $G \hookrightarrow GL(n, \overline{\mathbb{F}}_p)$  such that  $G$  is  $Frob_q$ -invariant, and  $\sigma^k$  is the restriction of the automorphism  $Frob_q$  to  $G$ . The fixpoint subgroup of  $\sigma$  is denoted by  $G^\sigma$ . We define  $q_\sigma = \sqrt[k]{q}$ .

*Remark 67.* The fixpoint set of  $Frob_q$  is clearly  $GL(n, \overline{\mathbb{F}}_p)^{Frob_q} = GL(n, \mathbb{F}_q)$ . More generally, if the closed subgroup  $G \leq GL(n, \overline{\mathbb{F}}_p)$  is  $Frob_q$ -invariant then  $G^{Frob_q} = G(\mathbb{F}_q)$ , the set of those elements whose matrix belongs to  $GL(n, \mathbb{F}_q)$ .

We will combine our previous results with the following powerful extension of the Lang-Weil estimates [31].

**Proposition 68** (Hrushovski). *Let  $G$  be a connected linear algebraic group and  $\sigma : G \rightarrow G$  a Frobenius map. Then there is a constant  $C = C(\dim(G), \deg(G))$  such that  $|G^\sigma|$  is approximately  $q_\sigma^{\dim(G)}$  with error*

$$\left| |G^\sigma| - q_\sigma^{\dim(G)} \right| \leq C \cdot q_\sigma^{\dim(G) - \frac{1}{2}}.$$

In the following corollary, besides various technical estimates, we establish that the finite group  $G^\sigma$  (if it is large enough) reflects the group-theoretic properties of  $G$ . E.g. there is a correspondence between subgroups of  $G$  and  $G^\sigma$ , and we have  $\mathcal{C}_G(G^\sigma) = \mathcal{Z}(G)$ .

**Corollary 69.** *For all parameters  $N > 0$ ,  $\Delta > 0$ ,  $I > 0$  and  $1 > \varepsilon > 0$  there is an integer  $K = K_L(N, \Delta, I, \varepsilon)$  with the following property.*

- (a) *Let  $G$  be a connected linear algebraic group,  $\sigma : G \rightarrow G$  a Frobenius map and  $\alpha \subseteq G^\sigma$  a finite subset. Suppose that  $\dim(G) \leq N$ ,  $\deg(G) \leq \Delta$ ,  $|G^\sigma : \langle \alpha \rangle| \leq I$  and  $|\alpha| \geq K$ . Then*

$$\dim(G) > 0, \quad \mathcal{C}_G(\alpha) = \mathcal{Z}(G), \quad \log(q_\sigma) \geq 1/\varepsilon.$$

- (b) *Let in addition  $A \leq G$  be a  $\sigma$ -invariant closed subgroup of degree  $\deg(A) \leq \Delta$ . Then  $A^\sigma = A \cap G^\sigma$ ,*

$$\left| \langle \alpha \rangle : \langle \alpha \rangle \cap A \right| \geq \frac{1-\varepsilon}{I\Delta} \left| G^\sigma \right|^{1-\dim(A)/\dim(G)} \geq \frac{1-\varepsilon}{I\Delta} \left| \langle \alpha \rangle \right|^{1-\dim(A)/\dim(G)}$$

*and if  $A \neq G$  then  $\langle \alpha \rangle \cap A \neq \langle \alpha \rangle$ .*

- (c) *Suppose furthermore that  $A$  is not normal in  $G$ . Then  $\alpha$  does not normalise  $A$  and*

$$(1 - \varepsilon) \log(q_\sigma) \leq \hat{\mu}(\langle \alpha \rangle, G, A) \leq (1 + \varepsilon) \log(q_\sigma).$$



*Proof.* Recall from Proposition 68 the constant  $C = C(N, \Delta)$ . By Proposition 68 we have

$$K \leq |\alpha| \leq |G^\sigma| \leq (1 + C)q_\sigma^N,$$

hence for large enough  $K$

$$\log(q_\sigma) \geq \log\left(\sqrt[N]{\frac{K}{1+C}}\right) > 1/\varepsilon$$

and  $\dim(G) > 0$  (see Remark 16). This proves the two inequalities of (a). In the rest of this proof we often use, that by choosing  $K$  large enough one can force  $q_\sigma$  to be arbitrary large.

It is obvious that  $A^\sigma = A \cap G^\sigma$ . By Proposition 68 for large enough  $q_\sigma$  (i.e. for large enough  $K$ ) we have

$$(1 - \frac{\varepsilon}{3})q_\sigma^{\dim(G)} \leq |G^\sigma| \leq (1 + \frac{\varepsilon}{3})q_\sigma^{\dim(G)}$$

and

$$(1 - \frac{\varepsilon}{3})q_\sigma^{\dim(A)} \leq |(A^0)^\sigma| \leq |A^\sigma| \leq \Delta |(A^0)^\sigma| \leq \Delta(1 + \frac{\varepsilon}{3})q_\sigma^{\dim(A)}.$$

Therefore

$$\begin{aligned} |G^\sigma : A^\sigma| &\geq \frac{(1 - \frac{\varepsilon}{3})q_\sigma^{\dim(G)}}{(1 + \frac{\varepsilon}{3})\Delta q_\sigma^{\dim(A)}} > \frac{1 - 2\frac{\varepsilon}{3}}{\Delta} q_\sigma^{\dim(G) - \dim(A)} > \\ &> \frac{1 - 2\frac{\varepsilon}{3}}{(1 + \frac{\varepsilon}{3})\Delta} |G^\sigma|^{1 - \dim(A)/\dim(G)} > \frac{1 - \varepsilon}{\Delta} |G^\sigma|^{1 - \dim(A)/\dim(G)}. \end{aligned}$$

This implies the inequality in (b). If  $A \neq G$  then  $\dim(A) < \dim(G)$ . Since  $|G^\sigma| \geq K$ , for large enough  $K$  we have  $|\langle \alpha \rangle : \langle \alpha \rangle \cap A| > 1$ , so  $\langle \alpha \rangle \neq \langle \alpha \rangle \cap A$ . This completes the proof of (b).

Let  $g \in \mathcal{C}_G(\alpha)$  be such that  $g \notin \mathcal{Z}(G)$ . Clearly all elements of the  $\langle \sigma \rangle$ -orbit  $g^{(\sigma)}$  commute with the elements of  $\alpha$ . On the other hand we know from (b) (say with parameter  $\varepsilon' = \frac{1}{2}$ ) that  $\langle \alpha \rangle \cap \mathcal{C}_G(g^{(\sigma)}) \neq \langle \alpha \rangle$ , which is a contradiction. Therefore  $\mathcal{C}_G(\alpha) = \mathcal{Z}(G)$  which completes the proof of (a).

Suppose now that  $A$  is not normal in  $G$ . We apply (b) (say with parameter  $\varepsilon'' = \frac{1}{2}$ ) to the proper subgroup  $\mathcal{N}_G(A) < G$ . We obtain that

$$\langle \alpha \rangle \neq \langle \alpha \rangle \cap \mathcal{N}_G(A) = \mathcal{N}_{\langle \alpha \rangle}(A)$$

i.e.  $\alpha$  does not normalise  $A$ .

By Fact 36 there is an upper bound  $\Delta' = \Delta'(N, \Delta) \geq \deg(\mathcal{N}_G(A))$ . We set  $\varepsilon''' = \frac{\varepsilon}{2N}$ . We apply (a) with a sufficiently small parameter  $\varepsilon'$  to obtain that  $\log(q_\sigma) > \frac{1}{\varepsilon'''} \left(1 + \log(\max(\Delta, \Delta', I))\right)$ . Let  $B \leq G$

be any  $\sigma$ -invariant closed subgroup with  $\dim(B) > 0$  and  $\deg(B) \leq \max(\Delta, \Delta')$ . We apply Proposition 68 to  $B^0$  and obtain

$$\left| \log |G^\sigma| - \dim(G) \cdot \log(q_\sigma) \right| < 1.$$

This gives us upper and lower bounds on  $\log |\langle \alpha \rangle \cap B|$ :

$$\begin{aligned} (1 - \varepsilon''') \dim(B) \log(q_\sigma) &\leq \\ &\leq \dim(B) \cdot \log(q_\sigma) - 1 - \log(I) \leq \log |(B^0)^\sigma| - \log(I) \leq \\ &\leq \log |\langle \alpha \rangle \cap B^0| \leq \log |\langle \alpha \rangle \cap B| \leq \log |B^\sigma| \leq \\ &\leq \log |(B^0)^\sigma| + \log(\max(\Delta, \Delta')) \leq \\ &\leq \dim(B) \cdot \log(q_\sigma) + 1 + \log(\max(\Delta, \Delta')) \leq \\ &\leq (1 + \varepsilon''') \dim(B) \log(q_\sigma) \end{aligned}$$

We apply these inequalities to  $B = G$  and to  $B = \mathcal{N}_G(A)$ :

$$(1 - \varepsilon''') \dim(G) \log(q_\sigma) \leq \log |\langle \alpha \rangle| \leq (1 + \varepsilon''') \dim(G) \log(q_\sigma)$$

and

$$\begin{aligned} (1 - \varepsilon''') \dim(\mathcal{N}_G(A)) \log(q_\sigma) &\leq \log |\mathcal{N}_{\langle \alpha \rangle}(A)| \leq \\ &\leq (1 + \varepsilon''') \dim(\mathcal{N}_G(A)) \log(q_\sigma). \end{aligned}$$

Subtracting the two estimates and dividing the result with  $\dim(G) - \dim(\mathcal{N}_G(A)) > 0$  we obtain

$$(1 - \varepsilon) \log(q_\sigma) \leq \frac{\log |\langle \alpha \rangle| - \log |\mathcal{N}_{\langle \alpha \rangle}(A)|}{\dim(G) - \dim(\mathcal{N}_G(A))} \leq (1 + \varepsilon) \log(q_\sigma)$$

and this completes the proof of (c).  $\square$

We arrived at a slightly more general version of Theorem 6 of the introduction:

**Theorem 70.** *For all parameters  $N > 0$ ,  $\Delta > 0$ ,  $I > 0$  and  $1 > \varepsilon > 0$  there is an integer  $M = M_{\text{main}}(N, \varepsilon)$  and a real  $K = K_{\text{main}}(N, \Delta, I, \varepsilon)$  with the following property.*

*Let  $G$  be a connected linear algebraic group over  $\overline{\mathbb{F}_p}$ . Let  $\sigma : G \rightarrow G$  a Frobenius map and  $1 \in \alpha \subseteq G^\sigma$  an ordered finite symmetric subset. Suppose that  $\mathcal{Z}(G)$  is finite,  $\dim(G) \leq N$ ,  $\deg(G) \leq \Delta$ ,  $\text{mult}(G) \leq \Delta$ ,  $\text{inv}(G) \leq \Delta$ ,  $|G^\sigma : \langle \alpha \rangle| \leq I$  and*

$$K \leq |\alpha| \leq q_\sigma^{(1-\varepsilon)\dim(G)}.$$

*Then there is a  $\sigma$ -invariant connected closed normal subgroup  $H \triangleleft G$  such that  $\deg H \leq K$ ,  $\dim(H) > 0$  and*

$$|\alpha^M \cap H| \geq |\alpha|^{(1+\delta)\dim(H)/\dim(G)}$$

where  $\delta = \frac{\varepsilon}{128N^3}$ . Moreover, our construction of the subgroup  $H$  is uniquely determined.

*Proof.* We set

$$M_{\text{CCC}} = M_{\text{CCC}}\left(N, \Delta, \frac{\varepsilon}{119N^3}\right), \quad M_s = M_s\left(N, \Delta, \frac{\varepsilon}{128N^3}\right), \\ M = M_{\text{CCC}} \cdot M_s,$$

$$K = \max\left(\Delta+1, K_{\text{CCC}}\left(N, \Delta, \frac{\varepsilon}{119N^3}\right), K_L\left(N, \Delta, I, \frac{\varepsilon}{3}\right), K_s\left(N, \Delta, \frac{\varepsilon}{128N^3}\right)\right).$$

By Corollary 69.(a)  $\dim(G) > 0$  and  $\mathcal{C}_G(\alpha) = \mathcal{Z}(G)$ , which is finite, hence  $\alpha|G$  is an  $(N, \Delta, K)$ -bounded spreading system. By assumption

$$\mu(\alpha, G) \leq (1 - \varepsilon) \log(q_\sigma).$$

Our construction of  $H$  will be uniquely determined, therefore it will be  $\sigma$ -invariant. By Corollary 69.(c) the rest of the conclusion of the theorem can be rewritten as follows.  $H$  is normalised by  $\alpha$ ,  $\deg(H) \leq K$ ,  $\dim(H) > 0$  and

$$\mu(\alpha^M, H) \geq (1 + \delta)\mu(\alpha, G)$$

i.e. we need to prove that  $\alpha|G$  is  $(\delta, M, K)$ -spreading and construct a subgroup of spreading that is uniquely determined.

If  $G$  were nilpotent then  $\mathcal{Z}(G)$  would have positive dimension. By assumption  $\mathcal{Z}(G)$  is finite, hence  $G$  is not nilpotent. We apply Lemma 65 with parameters  $N, \Delta$  and  $\frac{\varepsilon}{119N^3}$  to  $\alpha|G$ . In case we obtain a subgroup of spreading, the theorem holds. Otherwise we find a CCC-subgroup  $A \leq G$  which is not normal in  $G$  and satisfies

$$\mu(\alpha^{M_{\text{CCC}}}, A) > \left(1 - \frac{\varepsilon}{119N^3} \cdot 16N\right) \mu(\alpha, G) > \left(1 - \frac{1}{7N^2}\right) \left(1 + \frac{\varepsilon}{119N^2}\right) \mu(\alpha, G).$$

If  $\alpha|G$  is  $(\frac{\varepsilon}{119N^2}, M_{\text{CCC}}, K)$ -spreading, then it is  $(\delta, M, K)$ -spreading, the theorem holds in this case. So from now on we assume that

$$\mu(\alpha^{M_{\text{CCC}}}, G) < \left(1 + \frac{\varepsilon}{119N^2}\right) \mu(\alpha, G)$$

hence

$$\mu(\alpha^{M_{\text{CCC}}}, A) > \left(1 - \frac{1}{7N^2}\right) \mu(\alpha^{M_{\text{CCC}}}, G).$$

We know from Lemma 58 that  $\deg(A) \leq \deg(G)$  and Corollary 69.(c) with parameters  $N, \Delta, I$  and  $\frac{\varepsilon}{3}$  implies that

$$\begin{aligned} \hat{\mu}(\langle \alpha \rangle, G, A) &\geq \left(1 - \frac{\varepsilon}{3}\right) \log(q_\sigma) > \left(1 - \frac{\varepsilon}{2}\right) \left(1 + \frac{\varepsilon}{6}\right) \frac{\mu(\alpha, G)}{1 - \varepsilon} \geq \\ &\geq \frac{\left(1 - \frac{\varepsilon}{2}\right)}{1 - \varepsilon} \left(1 + \frac{\varepsilon}{119N^2}\right) \mu(\alpha, G) > \frac{\mu(\alpha^{M_{\text{CCC}}}, G)}{1 - \frac{\varepsilon}{2}}. \end{aligned}$$

We apply Lemma 63 with parameters  $N, \Delta$  and  $\frac{\varepsilon}{128N^3} = \delta$  to the spreading system  $\alpha^{M_{\text{CCC}}}|G$  and the subgroups  $W = G$  and  $A \leq G$ . If

we obtain a subgroup of  $(\delta, M_s, K)$ -spreading then it is a subgroup of  $(\delta, M, K)$ -spreading for  $\alpha|G$ , the theorem holds. Otherwise

$$\mu(\alpha^{M_{\text{ccc}}}, G) > (1 - \delta \cdot 64N^3) \hat{\mu}(\langle \alpha^{M_{\text{ccc}}} \rangle, G, A) = (1 - \frac{\varepsilon}{2}) \hat{\mu}(\langle \alpha \rangle, G, A),$$

a contradiction.  $\square$

*Remark 71.* In the proof of Theorem 4 one can avoid using Proposition 68. We know explicitly the number of elements in all finite simple groups of Lie type and also in their maximal tori (see e.g. [14]). When  $G$  is a connected adjoint simple algebraic group, one can show directly that  $(G^\sigma)'$  does not normalise any closed subgroup of positive dimension and small degree. This also implies that  $\mathcal{C}_G((G^\sigma)')$  is finite which is all we need for the proofs of Theorem 2 and Theorem 4.

The following result, communicated to us by Martin Liebeck, can be used to complete the above sketch. Let  $G$  be a connected adjoint simple algebraic group over an algebraically closed field  $\mathbb{F}$  of characteristic  $p$ , and  $\sigma$  a Frobenius morphism of  $G$ . Let  $G(q) = (G^\sigma)'$  and assume  $G(q)$  is simple.

**Proposition 72.** *There is no proper connected subgroup of  $G$  which contains  $G(q)$ .*

*Proof.* Suppose for a contradiction that  $G(q) < H < G$ , where  $H$  is connected.

First we consider the action of  $G(q)$  on the adjoint module  $L(G)$ . The  $G$ -composition factors of  $L(G)$  are well-known, and can be found in [46, 1.10]. With the exception of  $G = B_n, C_n, D_n, F_4$  with  $p = 2$  and  $G_2$  with  $p = 3$ ,  $G$  is either irreducible on  $L(G)$ , or has two composition factors, one of which is trivial. In any case, each composition factor is either a restricted  $\mathbb{F}G$ -module, or a field twist of one. It follows that  $G(q)$  is irreducible on every  $G$ -composition factor of  $L(G)$ . Therefore  $H$  is also irreducible on every  $G$ -composition factor of  $L(G)$ , and hence  $H$  must be a semisimple group.

For the moment exclude the exceptions  $B_n, \dots, G_2$  in the above paragraph. Clearly  $G(q)$  fixes  $L(H) \subset L(G)$ , so it follows that  $L(H)$  must be a composition factor of co-dimension 1 in  $L(G)$ . If  $U_H$  is a maximal connected unipotent subgroup of  $H$ , then a standard result tells us that  $\dim H = 2 \dim U_H + \text{rank}(H)$ . Since  $\dim H = \dim G - 1$ , it follows that  $U_H$  is also a maximal unipotent subgroup of  $G$ , and  $\text{rank}(H) = \text{rank}(G) - 1$ . So the root system of  $H$  has the same number of roots as that of  $G$ , and  $H$  has rank 1 less than  $G$ . An easy check of root systems shows that this is impossible.

It remains to handle the exceptional cases  $G = B_n, C_n, D_n, F_4$  ( $p = 2$ ) and  $G_2$  ( $p = 3$ ). Consider  $G_2$  and  $F_4$ , and let  $H_0$  be a simple factor of  $H$  which contains an isomorphic copy of  $G(q)$ . Then  $H_0$  is of rank at most 2 (resp. 4), and the smallest projective representation of  $H_0$  has dimension at least that of  $G(q)$ , which is 7 (resp. 26). This is clearly impossible.

Next let  $G = D_n$ . Here the  $G$ -composition factors of  $L(G)$  are of high weights  $\lambda_2, 0$  ( $n$  odd) or  $\lambda_2, 0^2$  ( $n$  even). We have already dealt with the case where  $\dim H = \dim G - 1$ , so we may assume  $n$  is even and  $\dim H = \dim G - 2$ . Then either  $\dim U_H = \dim U_G$ ,  $\text{rank}(H) = \text{rank}(G) - 2$ , or  $\dim U_H = \dim U_G - 1$ ,  $\text{rank}(H) = \text{rank}(G)$ . An inspection of root systems shows that neither of these is possible.

Now let  $G = C_n$ , and let  $V$  be the natural  $2n$ -dimensional  $G$ -module. As  $G(q)$  cannot act nontrivially on a module of dimension less than  $2n$ , it must act tensor indecomposably on  $V$ , and hence so does  $H$ . Therefore  $H$  is simple. The possibilities for  $G(q)$  are  $C_n(q)$  and  $Sz(q)$  (the latter just for  $n = 2$ ). In the former case  $G(q)$  has an elementary abelian subgroup  $R = r^n$ , where  $r$  is a prime dividing  $q + 1$ . Note that  $r$  is odd as  $p = 2$ . Also  $\text{rank}(H) \leq \text{rank}(G) = n$ . An elementary argument (see [16, Section 2]) shows that the abelian  $r$ -rank of  $H$  is equal to  $\text{rank}(H)$ , and hence  $\text{rank}(H) = n$ . The only possibility is that  $H = D_n$ . But  $G(q) = C_n(q)$  does not lie in  $D_n$  as it does not fix a quadratic form on  $V$ . If  $G(q) = Sz(q)$  then  $H$  cannot have rank 2 (as  $C_2$  has no connected simple proper subgroup of rank 2), so  $H = A_1$ ; but  $Sz(q) \not\leq A_1$ , a contradiction.

Finally, if  $G = B_n$  then there is a morphism from  $G$  to  $C_n$  which is an isomorphism of abstract groups, and applying this morphism to  $G(q)$  and  $H$ , we reduce to the  $C_n$  case. This completes the proof.  $\square$

## 12. LINEAR GROUPS OVER FINITE FIELDS

In this section we first prove our main theorem concerning simple groups of Lie type and various results for  $p$ -generated subgroups of  $GL(n, \mathbb{F}_p)$  i.e. subgroups generated by elements of order  $p$ . These finite groups can be obtained roughly as fixpoint groups of Frobenius maps of linear algebraic groups. Theorem 4 is essentially a special case of Theorem 70. For perfect  $p$ -generated groups Theorem 7 follows by an inductive argument based on Theorem 70. To prove Theorem 7 in the general case we need a number of finite group-theoretic results.

For the following useful results see [52] and [29, proof of Lemma 2.2].

**Proposition 73** (Olson). *Let  $1 \in \alpha$  be a generating set of a finite group  $G$  and  $\beta$  a nonempty subset of  $G$ . Then  $|\alpha\beta| \geq \min(|\beta| + |\alpha|/2, |G|)$ . In particular, if  $\alpha^3 \neq G$  then  $|\alpha^3| \geq 2|\alpha|$ .*

□

As noted in [30] the following proposition is essentially due to Ruzsa (see [55] and [56]).

**Proposition 74.** *Let  $\alpha$  be a finite subset of a group. Then*

a)

$$\frac{|(\alpha \cup \alpha^{-1} \cup \{1\})^3|}{|\alpha|} \leq \left(3 \frac{|\alpha^3|}{|\alpha|}\right)^3$$

b) *If  $\alpha = \alpha^{-1}$  is a symmetric set with  $1 \in \alpha$  and  $m \geq 2$  an integer then*

$$\frac{|\alpha^m|}{|\alpha|} \leq \left(\frac{|\alpha^3|}{|\alpha|}\right)^{m-2}$$

□

As mentioned in the introduction, a result of Gowers [26] implies the following.

**Proposition 75** (Nikolov, Pyber [50]). *Let  $G$  be a finite group and let  $k$  denote the minimal degree of a complex representation. Suppose that  $\alpha$ ,  $\beta$  and  $\gamma$  are subsets of  $G$  such that*

$$|\alpha||\beta||\gamma| > \frac{|G|^3}{k}.$$

*Then  $\alpha\beta\gamma = G$ . In particular, if  $|\alpha| > |G|/\sqrt[3]{k}$  then  $\alpha^3 = G$ .*

**Proposition 76.** *Let  $G$  be a simple algebraic group and  $\sigma : G \rightarrow G$  a Frobenius map. If  $L$  is the simple group of Lie type obtained as a composition factor of  $G^\sigma$  then the minimal degree of a complex representation of  $L$  is at least  $\frac{q_\sigma - 1}{2}$ . If  $q_\sigma \geq 20$  and  $\alpha \subseteq L$  is a subset of size at least  $q_\sigma^{\dim(G) - \frac{1}{4}}$  then  $\alpha^3 = L$ .*

*Proof.* The first statement is an obvious consequence of the Landazuri-Seitz lower bounds ([45] cf. [38, Table 5.3A]). If  $q_\sigma \geq 4$  then  $|L| \leq q_\sigma^{\dim(G)}$  (see [15]). Now the second statement follows from Proposition 75. □

We are now ready to prove our main result, Theorem 4.

**Theorem 77.** *For all parameters  $r > 0$  there is a real  $\varepsilon = \varepsilon(r) > 0$  with the following property.*

*Let  $L$  be a finite simple group of Lie type of Lie rank at most  $r$  and  $\alpha \subset L$  a generating set. Then either  $\alpha^3 = L$  or*

$$|\alpha^3| \geq |\alpha|^{1+\varepsilon}.$$

*Proof.* There is a simple adjoint algebraic group  $G$  and a Frobenius map  $\sigma : G \rightarrow G$  such that  $L \leq G^\sigma$ , and there are universal bounds  $I(r)$ ,  $N(r)$  and  $\Delta(r)$  such that

$$|G^\sigma : L| \leq I(r), \quad \dim(G) \leq N(r),$$

$$\deg(G) \leq \Delta(r), \quad \text{mult}(G) \leq \Delta(r), \quad \text{inv}(G) \leq \Delta(r).$$

If  $|\alpha| \geq q_\sigma^{\dim(G)-\frac{1}{4}}$  and  $q_\sigma \geq 20$  then  $\alpha^3 = L$  by Proposition 76. Assume otherwise.

Suppose first that  $\alpha = \alpha^{-1}$  is symmetric with  $1 \in \alpha$ . We apply Theorem 70 with parameters  $N(r)$ ,  $\Delta(r)$ ,  $I(r)$  and  $\varepsilon' = \frac{1}{4\dim(G)}$  and obtain an integer  $M = M(r)$  and a real  $K = K(r)$ . We may assume that  $M \geq 3$ , and by Corollary 69.(a) we may increase  $K$  so that  $|\alpha| \geq K$  implies  $q_\sigma \geq 20$ . Since  $G$  is simple, we have  $G = H$  now. If  $K \leq |\alpha| \leq q_\sigma^{\dim(G)-\frac{1}{4}}$  then by Theorem 70 we have

$$|\alpha^M| \geq |\alpha|^{1+\frac{1}{512N^4}}.$$

Finally we assume  $|\alpha| \leq K$  and  $\alpha^3 \neq L$ . By Proposition 73 we have

$$|\alpha^3| \geq 2|\alpha| \geq |\alpha|^{1+\varepsilon''}$$

where  $\varepsilon'' = \min\left(\frac{\log(2)}{\log(K)}, \frac{1}{512N^4}\right)$  (which depends only on  $r$ ). We obtain that in any case

$$|\alpha^M| \geq |\alpha|^{1+\varepsilon''}.$$

The theorem follows in the symmetric case from Proposition 74.(b).

The general case then follows using Proposition 74.(a).  $\square$

In Theorem 70 it is essential to assume that the centre of the algebraic group  $G$  is finite. Without this assumption the statement fails. However, we can complement it for finite groups with possibly large centre using the following special case of a deep result of Nikolov and Segal ([51, Theorem 1.7]).

**Proposition 78.** *Let  $P$  be a finite perfect group generated by  $d$  elements. Then every element of  $G$  is the product of  $g(d)$  commutators where  $g(d) = 12d^3 + \mathcal{O}(d^2)$  depends only on  $d$ .*

Next we will describe more precisely the Nori correspondence between  $p$ -generated subgroups of  $GL(n, \mathbb{F}_p)$  and certain closed subgroups of  $GL(n, \overline{\mathbb{F}}_p)$  and some other useful facts about perfect  $p$ -generated subgroups.

**Proposition 79.** *Let  $P \leq GL(n, \mathbb{F}_p)$  be a  $p$ -generated subgroup. Then there are bounds  $I = I_{\exp}(n)$ ,  $\Delta = \Delta_{\exp}(n)$  and  $K = K_{\exp}(n)$  with the following properties.*

- (a) *There is a  $Frob_p$ -invariant connected closed subgroup  $G \leq GL(n, \overline{\mathbb{F}}_p)$  such that  $\dim(G) \leq n^2$ ,  $\deg(G) \leq \Delta$ ,  $\text{mult}(G) \leq \Delta$ ,  $\text{inv}(G) \leq \Delta$  and  $P$  is a subgroup of  $G(\mathbb{F}_p)$  of index at most  $I$ .*
- (b) *If  $P$  is perfect then the degree of any complex representation is at least  $(p-1)/2$ .*
- (c) *If moreover  $|P| \geq K$  and  $\alpha \subseteq P$  is a subset of size  $|\alpha| \geq p^{\dim(G)-\frac{1}{4}}$  then  $\alpha^3 = P$ .*

*Proof.* We first prove (a). By a result of Nori [49] there is a constant  $I = I_{\exp}(n)$  such that there is a  $Frob_p$ -invariant connected closed subgroup  $G \leq GL(n, \overline{\mathbb{F}}_p)$  with  $P \leq G(\mathbb{F}_p)$  of index  $|G(\mathbb{F}_p) : P| \leq I$ . Clearly  $\dim(G) \leq n^2$ . By [41, Proposition 3] there is an upper bound  $\Delta_{\exp}(n) \geq \deg(G)$  (which can also be proved easily from [49] using the degree of the exponential map) and by Proposition 31 we can also assume that it is also an upper bound on the other numerical invariants  $\text{mult}(G)$  and  $\text{inv}(G)$ . Let  $\sigma : G \rightarrow G$  denote the restriction to  $G$  of the automorphism  $Frob_p : G \rightarrow G$  of Definition 66, then  $G(\mathbb{F}_p) = G^\sigma$  by Remark 67.

Assume now that  $P$  is perfect. Let  $\phi : P \rightarrow GL(k, \mathbb{C})$  be a nontrivial complex representation. If  $k < \frac{p-1}{2}$  then by well-known results of Brauer and Feit-Thompson (see e.g. [36, Theorem 14.11] and the remark after its proof)  $\phi(P)$  has a normal Sylow- $p$  subgroup. This is impossible since  $\phi(P)$  is also a perfect  $p$ -generated group. This proves (b).

If  $K$  is large enough then  $p \geq K^{1/n^2}$  is large as well, hence by Proposition 68 we have  $|P| \leq 2p^{\dim(G)}$  and  $\alpha^3 = P$  by Proposition 75.  $\square$

**Proposition 80.** *Let  $H \leq GL(n, \overline{\mathbb{F}})$  be a closed subgroup. Then for some  $n' = n'(n, \deg(H))$  there is a homomorphism  $\phi_H : \mathcal{N}_{GL(n, \overline{\mathbb{F}})}(H) \rightarrow GL(n', \overline{\mathbb{F}})$  of degree bounded by  $n$  and  $\deg(H)$  whose kernel is  $H$ . Moreover, if  $\overline{\mathbb{F}}$  has characteristic  $p$  and  $H$  is  $Frob_q$ -invariant for some  $p$ -power  $q$  then the homomorphism  $\phi_H$  we construct is  $Frob_q$ -equivariant (see Definition 66.(b)).*

This proposition is a mild strengthening of [34, Theorem 11.5], and it is rather clear that the proof can easily be modified to yield this version. Since we did not find a good reference, we reproduce here



the argument. The modified proof is based on the notion of *families of subgroups*, we recall the definition and prove some of their basic properties.

Throughout the proof the adjectives ( $Frob_q$ -invariant) and ( $Frob_q$ -equivariant) appearing in parenthesis apply only in the case when  $\mathcal{H}$  is  $Frob_q$ -invariant.

**Definition 81.** To simplify the notation let  $G = GL(n, \overline{\mathbb{F}})$ . Suppose that  $T$  is an affine algebraic set and  $\mathcal{H} \subseteq T \times G$  is a closed subset. As in [34], let  $K[G]$  and  $K[T \times G]$  denote the coordinate rings of  $G$  and  $T \times G$  respectively. For each point  $t \in T$  we consider the closed subset  $\mathcal{H}_t \subseteq G$  defined via the equation  $\{t\} \times \mathcal{H}_t = \mathcal{H} \cap (\{t\} \times G)$ . We call  $\mathcal{H}$  a *family of subgroups* if  $\mathcal{H}_t$  is a subgroup of  $G$  for each  $t \in T$ . In this case we call  $T$  the *parameter space* and  $\mathcal{H}_t$  are the *members* of the family. Similarly, for vectorspaces  $V$  and  $W$ , a closed subset  $\mathcal{M} \subseteq T \times W$  is a *family of subspaces* if each  $\mathcal{M}_t \subseteq W$  is a subspace of  $W$ , and a closed subset  $\mathcal{L} \subseteq T \times V$  is called a *family of lines* if each  $\mathcal{L}_t \subseteq V$  is a line through the origin. A morphism from a family of subgroups  $\mathcal{H}$  of  $GL(n, \overline{\mathbb{F}})$  to another group  $GL(m, \overline{\mathbb{F}})$  is a *family of homomorphisms* if the induced morphisms  $\mathcal{H}_t \rightarrow GL(m, \overline{\mathbb{F}})$  are all homomorphisms.

**Claim 82.** *Let  $T$  be an affine algebraic set and  $F < K[T \times G]$  a finite dimensional subspace. Then the smallest  $G$ -invariant subspace  $W < K[T \times G]$  containing  $F$  is finite dimensional. Moreover, if  $T$  and  $F$  are  $Frob_q$ -invariant then  $W$  is also  $Frob_q$ -invariant.*

*Proof.*  $G$  acts on  $T \times G$  via the right multiplication in the second factor. Then  $W$  is finite dimensional by [34, Proposition 8.6], and the  $Frob_q$ -invariance is obvious.  $\square$

**Claim 83.** *Let  $\mathcal{H} \subseteq T \times G$  be a family of subgroups. Then there is a rational representation  $\psi : G \rightarrow GL(V)$ , a dense open subset  $U \subseteq T$  and a family of lines  $\mathcal{L} \subset U \times V$  such that*

$$\mathcal{H}_t = \{g \in G \mid \psi(g)\mathcal{L}_t = \mathcal{L}_t\}$$

*for all  $t \in U$ . Moreover, if  $\mathcal{H}$  is  $Frob_q$ -invariant then our construction yields  $Frob_q$ -invariant  $\psi$ ,  $U$  and  $\mathcal{L}$ .*

*Proof.* We shall imitate [34, proof of 11.2]. Let  $I \triangleleft K[T \times G]$  denote the ideal of  $\mathcal{H}$  (i.e. the set of those functions vanishing on  $\mathcal{H}$ ) and  $I_t \triangleleft K[G]$  for  $t \in T$  the ideal of  $\mathcal{H}_t$ . Then  $I$  is generated by a ( $Frob_q$ -invariant) finite dimensional subspace  $F \leq K[T \times G]$ . By Claim 82 there is a finite dimensional  $G$ -invariant subspace  $W < K[T \times G]$  containing  $F$  (which is also  $Frob_q$ -invariant). For each  $t \in T$  the restriction of functions to  $\{t\} \times G$  is a ring homomorphism  $r_t : K[T \times G] \rightarrow K[G]$ .

The closed subset of  $G$  corresponding to the ideal  $r_t(I)$  is precisely  $\mathcal{H}_t$ , but the ideal  $r_t(I)$  may not be a radical ideal, hence it is not necessarily equal to  $I_t$ . It is folklore that there is a ( $Frob_p$ -invariant) dense open subset  $T^* \subseteq T$  such that  $r_t(I) = I_t$  for all  $t \in T^*$ . Here is a quick sketch. We consider the projection morphism  $\pi : \mathcal{H} \rightarrow T$ . By [39, Theorem I.1.6] there is a canonical open dense subset  $T'$  such that the restriction  $\pi^{-1}(T') \rightarrow T'$  is flat. The fibre of  $\pi$  at the generic points of  $T'$  are smooth varieties (i.e. closed subgroups), hence by [27, Exercise III/10.2] there is a canonical open dense subset  $T^* \subseteq T'$  such that the restriction  $\pi^{-1}(T^*) \rightarrow T^*$  is smooth. By [27, Theorem III/10.2] the rings  $K[G]/r_t(I)$  are regular for all  $t \in T^*$ . In particular,  $r_t(I)$  are radical ideals, hence  $r_t(I) = I_t$  for all  $t \in T^*$ .

We set  $\mathcal{M}_t = W \cap r_t^{-1}(I_t)$ . Then  $\mathcal{M} = \bigcup_t \{t\} \times \mathcal{M}_t \subseteq T^* \times W$  is a family of subspaces, hence the function  $t \rightarrow \dim(\mathcal{M}_t)$  is an upper semi-continuous function on  $T^*$ . Let  $T^* = \bigcup_i T_i^*$  be the irreducible decomposition of  $T^*$  and  $d_i = \max_{t \in T_i^*} \dim(\mathcal{M}_t)$ . The set of points  $t \in T_i^*$  which satisfy  $\dim(\mathcal{M}_t) = d_i$  form an open dense subset  $U_i \subseteq T_i^*$ . Then  $U = \bigcup_i U_i$  is a ( $Frob_q$ -invariant) open dense subset of  $T$ . We set  $V = \bigoplus_{j=0}^{\dim(W)} \bigwedge^j W$  and the representation  $\psi : G \rightarrow GL(V)$  is just the natural  $G$ -action on  $V$ . For  $t \in U_i$  we set  $\mathcal{L}_t = \bigwedge^{d_i} \mathcal{M}_t \leq \bigwedge^{d_i} W \leq V$  and let  $\psi_t : G \rightarrow GL(r_t(W))$  be the natural  $G$ -action on  $r_t(W)$ .

Then  $\mathcal{L} = \bigcup_{t \in U} \{t\} \times \mathcal{L}_t \subseteq U \times V$  is a ( $Frob_q$ -invariant) family of lines and for each  $t \in U$  the stabiliser of  $\mathcal{L}_t$  in  $\psi(G)$  is equal to the stabiliser of  $\mathcal{M}_t$  in the image of  $G$  in  $GL(W)$ , which is in turn equal to the stabiliser of  $r_t(M_t) = I_t \cap r_t(W)$  in  $\psi_t(G)$ . On the other hand this last stabiliser is just  $\mathcal{H}_t$  by [34, proof of 11.2].  $\square$

**Claim 84.** *Let  $\mathcal{H} \subseteq T \times G$  be a family of subgroups. Then there is a family of homomorphisms  $\phi : \mathcal{N}_G(\mathcal{H}_t) \rightarrow GL(n', \overline{\mathbb{F}})$  for a common value of  $n'$ . In particular, there is a common upper bound on  $\deg(\phi_t)$ . Moreover, if  $\mathcal{H}$  is  $Frob_q$ -invariant then our construction yields a  $Frob_q$ -equivariant  $\phi$  (see Definition 66.(b)).*

*Proof.* We prove the claim by induction on  $\dim(T)$ . We apply Claim 83 (and use its notation) to this family of subgroups. We obtain an open dense subset  $U \subseteq T$ . Then  $\dim(T \setminus U) < \dim(T)$  so by the induction hypothesis for each ( $Frob_q$ -invariant)  $t \in T \setminus U$  there is a ( $Frob_q$ -equivariant) embedding  $\mathcal{N}_G(\mathcal{H}_t)/\mathcal{H}_t \rightarrow GL(n'', \overline{\mathbb{F}})$  with a common  $n''$  and a common bound on their degrees.

Consider any ( $Frob_q$ -invariant) point  $t \in U$  and apply [34, proof of Theorem 11.5] to the subgroup  $N = \mathcal{H}_t$  of  $\mathcal{N}_G(\mathcal{H}_t)$  (which is denoted there by  $G$ ). For the representation and the line at the beginning of that

proof we may choose our  $G \rightarrow GL(V)$  and  $\mathcal{L}_t \leq V$ . The proof then constructs a representation  $\phi_{\mathcal{H}_t} : \mathcal{N}_G(\mathcal{H}_t) \rightarrow GL(W)$  whose kernel is just  $\mathcal{H}_t$ . Moreover, the homomorphisms  $\phi_{\mathcal{H}_t}$  together form a family of homomorphisms  $G \times T \rightarrow GL(W)$ , hence there is a common upper bound on their degrees. The construction is uniquely determined, so it must be  $Frob_q$ -equivariant whenever  $\mathcal{H}$  and  $t$  are so. Moreover, by construction  $\dim(W) \leq \dim(V)^2$ , hence the Claim is valid with  $n' = \max(n'', \dim(V)^2)$ .  $\square$

*Proof of Proposition 80.* By [39, Section I.3] there is a canonical open subset of the Chow variety of the projectivisation of  $G$  which parametrises all the closed subgroups of  $G$  of degree  $\deg(H)$ . This open subset is not necessarily affine, but it is defined over  $\mathbb{F}_q$ , hence it is the union of finitely many  $Frob_q$ -invariant affine subvarieties. Hence there is a  $Frob_q$ -invariant family of subgroups which contains (as members) all the closed subgroups of  $G$  of degree  $\deg(H)$ . The proposition follows from Claim 84 applied to this family.  $\square$

The proofs of all the results obtained in this section concerning not necessarily simple subgroups of  $GL(n, \mathbb{F}_p)$  rest on the following somewhat technical consequence of Theorem 70. This theorem complements the results about growth of generating sets of simple groups. It would be most interesting to establish an appropriate analogue for subgroups of  $GL(n, \mathbb{F}_q)$ .

**Theorem 85.** *For all parameters  $n > 0$  there is a real  $\varepsilon = \varepsilon(n) > 0$  with the following property.*

*Let  $P \leq GL(n, \mathbb{F}_p)$  be a perfect  $p$ -generated subgroup. Let  $1 \in \alpha \subseteq P$  be a symmetric generating set which projects onto each simple quotient of  $P$ . Then either  $\alpha^3 = P$  or*

$$|\alpha^3| \geq |\alpha|^{1+\varepsilon}.$$

*Moreover, the diameter of the Cayley graph of  $P$  with respect to  $\alpha$  is at most  $d(n)$  where  $d(n)$  depends on  $n$ .*

*Proof.* Let  $l$  be the smallest integer such that  $|P| \leq p^{l/2}$ , note that  $l \leq 2n^2$ . We prove the first statement (concerning  $\alpha^3$ ) by induction on  $l$ . For  $l = 0$  it is clear. We assume that  $l > 0$  and the statement holds for all groups of order at most  $p^{(l-1)/2}$  and for all matrix sizes  $n$  with an  $\varepsilon$ -value  $\varepsilon'(n, l) \leq 1$ .

We apply Proposition 79 to  $P$  and obtain the bounds  $I_{\exp}, \Delta_{\exp}, K_{\exp}$  (which depend only on  $n$ ) and the  $Frob_p$ -invariant connected closed subgroup  $G \leq GL(n, \overline{\mathbb{F}_p})$  for which  $|G(\mathbb{F}_p) : P| \leq I_{\exp}$  and  $\dim(G) \leq$

$n^2$ . We shall apply Theorem 70 with parameter  $\varepsilon'' = \frac{1}{4\dim(G)}$  and obtain the constants

$$\delta = \frac{\varepsilon''}{128\dim(G)^3}, \quad M_{\text{main}} = M_{\text{main}}(\dim(G), \varepsilon''),$$

$$K_{\text{main}} = K_{\text{main}}(\dim(G), \Delta_{\text{exp}}, I_{\text{exp}}, \varepsilon'').$$

We shall choose later a real  $K \geq \max(K_{\text{main}}, K_{\text{exp}})$ . If  $|\alpha| \leq K$  and  $\alpha^3 \neq P$  then  $|\alpha^3| \geq 2|\alpha|$  by Proposition 73 and the induction step is complete in this case with any  $\varepsilon \geq \log(2)/\log(K)$ . So we may assume that  $|\alpha| > K$ . If  $|\alpha| > p^{\dim(G)-1/4}$  then  $\alpha^3 = P$  by Proposition 79.(c). So we assume

$$K < |\alpha| \leq p^{\dim(G)-\frac{1}{4}}.$$

Consider all  $Frob_p$ -invariant connected closed normal subgroups  $1 \neq H \triangleleft G$  of degree  $\deg(H) \leq K_{\text{main}}$ . Then by Proposition 69.(b), for sufficiently large  $K$  either  $H = G$  or  $\alpha \not\subseteq H$ . By Proposition 80 there is a  $Frob_p$ -equivariant homomorphism  $G \rightarrow GL(n', \mathbb{F}_p)$  for some common  $n' = n'(\dim(G), K_{\text{main}})$  whose kernel is  $H$ . The elements of  $\alpha$  are fixpoints of  $Frob_p$ , so by the equivariance their images are also fixpoints of  $Frob_p$  (see Definition 66.(b)), i.e. the image set  $\alpha_H$  of  $\alpha$  generates a subgroup of  $GL(n', \mathbb{F}_p)$  isomorphic to  $P/(H \cap P)$ . This subgroup is again perfect,  $p$ -generated and  $\alpha_H$  projects onto each of its simple quotients. In particular, if  $H \neq G$  i.e.  $\alpha_H \neq \{1\}$  then  $|\alpha_H| \geq p \geq |\alpha|^{1/n^2}$ . We know from Proposition 68 that if  $K$  is large enough then  $|H \cap P| \geq |H(\mathbb{F}_p)|/I_{\text{exp}} > \sqrt{p}$  so  $|P/(H \cap P)| < |P|/\sqrt{p} \leq p^{(l-1)/2}$  and the induction hypothesis holds for  $\alpha_H$  and  $P/(H \cap P)$  with the  $\varepsilon$ -value  $\varepsilon' = \varepsilon'(n', l) \leq 1$ .

Suppose that we find such an  $H$  different from  $G$  and  $|\alpha_H^3| \geq |\alpha_H|^{1+\varepsilon'}$ . Then using Proposition 43 we obtain

$$|\alpha^5| \geq |\alpha_H^3| \cdot |\alpha^2 \cap H| \geq |\alpha_H|^{1+\varepsilon'} \cdot |\alpha^2 \cap H| \geq |\alpha| \cdot |\alpha_H|^{\varepsilon'} \geq |\alpha|^{1+\varepsilon'/n^2}$$

and by Proposition 74.(b) the induction step is complete. So we may assume that for all such  $H$  we have  $\alpha_H^3 = P/(H \cap P)$ . It follows from Corollary 69.(b) that if  $K$  is sufficiently large then

$$|\alpha_H^3| = |P/(H \cap P)| \geq |P|^{1-\dim(H)/\dim(G)-\delta/(2n^2)} \geq |\alpha|^{1-\dim(H)/\dim(G)-\delta/(2n^2)}.$$

Suppose next that  $\mathcal{Z}(G)$  is finite. We apply Theorem 70 with parameters  $\dim(G)$ ,  $\Delta_{\text{exp}}$ ,  $I_{\text{exp}}$  and  $\varepsilon'' = \frac{1}{4\dim(G)}$  to the subset  $\alpha \subset G^{Frob_p}$ . We obtain a  $Frob_p$ -invariant connected closed normal subgroup  $H \triangleleft G$  such that  $\deg(H) \leq K_{\text{main}}$ ,  $\dim(H) > 0$  and

$$|\alpha^{M_{\text{main}}} \cap H| \geq |\alpha|^{(1+\delta)\dim(H)/\dim(G)}.$$

If  $H = G$  then  $|\alpha^{M_{\text{main}}}| \geq |\alpha|^{(1+\delta)}$ , otherwise

$$|\alpha^{3+M_{\text{main}}}| \geq |\alpha_H^3| \cdot |\alpha^{M_{\text{main}} \cap H}| \geq |\alpha|^{1 - \frac{\dim(H)}{\dim(G)} - \frac{\delta}{2n^2}} \cdot |\alpha|^{(1+\delta) \frac{\dim(H)}{\dim(G)}} \geq |\alpha|^{1 + \frac{\delta}{2n^2}}.$$

By Proposition 74.(b) the induction step is complete in this case as well.

Finally we suppose that  $\mathcal{Z}(G)$  is infinite. In this case we consider the normal subgroup  $H = \mathcal{Z}(G)^0$ . By assumption  $\alpha_H^3 = P/(H \cap P)$  hence  $\alpha^3$  intersects every  $(H \cap P)$ -coset in  $P$ . Hence every commutator element of  $P$  is in fact the commutator of two elements in  $\alpha^3$ . It is well-known that  $P$  is generated by at most  $n^2$  elements (see [54]) hence by Proposition 78 each element of  $P$  is the product of  $Cn^6$  commutators for some constant  $C$ . By assumption  $|\alpha| \leq p^{\dim(G)-1/4}$ . Since  $|P| \geq |\alpha| > K$ , if we choose  $K$  sufficiently large then  $|P| \geq p^{\dim(G)-1/8}$  by Proposition 68. Therefore

$$|\alpha^{3 \cdot 4 \cdot Cn^6}| = |P| > |\alpha|^{1+1/8 \dim(G)}$$

and by Proposition 74.(b) the induction step is complete in this case too. The first statement is proved.

Let us apply the (now established) first statement successively to  $\alpha, \alpha^3, \alpha^9, \dots$ . We obtain by induction that either  $\alpha^{3^i} = P$  or  $|\alpha^{3^i}| \geq |\alpha|^{(1+\varepsilon)^i}$  for all  $i$ . By assumption  $|\alpha| \geq p$  and  $|P| < p^{n^2}$  hence  $\alpha^{d(n)} = P$  where  $d(n)$  is the smallest integer above  $n^{2 \log(3)/\log(1+\varepsilon)}$ . That is, the diameter of the Cayley graph with respect to  $\alpha$  is at most  $d(n)$ .  $\square$

Now we prove Theorem 8 of the Introduction.

**Theorem 86.** *For all natural numbers  $n$  there is an integer  $M = M(n)$  with the following property.*

*Let  $P \leq GL(n, \mathbb{F}_p)$  be a perfect  $p$ -generated subgroup. Then the diameter of the Cayley graph of  $P$  with respect to any symmetric generating set is at most  $(\log |P|)^M$ .*

*Proof.* Let  $\alpha$  be a symmetric generating set of  $P$  containing 1. Let  $L$  be any simple quotient of  $P$ , we denote by  $\tilde{\alpha}$  the image of  $\alpha$  in  $L$ . The Lie rank of  $L$  is at most  $n$  (see [21] and [38, Proposition 5.2.12]). Let  $\varepsilon = \varepsilon(n)$  be as in Theorem 77. Applying that theorem successively to  $\tilde{\alpha}, \tilde{\alpha}^3, \tilde{\alpha}^9, \dots$  we obtain by induction that either  $\tilde{\alpha}^{3^i} = L$  or  $|\tilde{\alpha}^{3^i}| \geq |\tilde{\alpha}|^{(1+\varepsilon)^i} \geq 3^{(1+\varepsilon)^i}$  for all  $i$ . With  $m = \frac{\log \log |P| - \log \log(3)}{\log(1+\varepsilon)}$  we obtain that  $|\tilde{\alpha}^{3^m}| \geq |P| \geq |L|$  hence  $\alpha^{3^m}$  projects onto  $L$ . This holds for each simple quotient with the same exponent  $m$ .

By Theorem 85 the diameter of the Cayley graph corresponding to  $\alpha^{3^m}$  is at most  $d(n)$ , hence the diameter of the Cayley graph corresponding to  $\alpha$  is at most  $3^m d(n) \leq (\log |P|)^{M(n)}$  where  $M(n)$  is the smallest integer above  $\frac{\log(3)}{\log(1+\varepsilon)} + \log(d(n))$   $\square$

We will reduce the proof of Theorem 7 to the perfect  $p$ -generated case (more precisely to Theorem 85) using finite group theory.

**Definition 87.** As usual  $Sol(G)$  denotes the soluble radical and  $O_p(G)$  the maximal normal  $p$ -subgroup of a finite group  $G$ . A group is called *quasi-simple* if it is perfect and simple modulo its centre. We denote by  $Lie^*(p)$  the set of direct products of simple groups of Lie type of characteristic  $p$ , and by  $Lie^{**}(p)$  the set of central products of quasi-simple groups of Lie type of characteristic  $p$ . If  $G/Sol(G)$  is in  $Lie^*(p)$  then we call  $G$  a *soluble by  $Lie^*(p)$  group*.

The following deep result is essentially due to Weisfeiler [61].

**Proposition 88.** *Let  $G$  be a finite subgroup of  $GL(n, \mathbb{F})$  where  $\mathbb{F}$  is a field of characteristic  $p > 0$ . Then  $G$  has a normal subgroup  $H$  of index at most  $f(n)$  such that  $H \geq O_p(G)$  and  $H/O_p(G)$  is the central product of an abelian  $p'$ -group and quasi-simple groups of Lie type of characteristic  $p$ , where the bound  $f(n)$  depends on  $n$ .*

It was proved by Collins [17] that for  $n \geq 71$  one can take  $f(n) = (n+2)!$ . Remarkably a (non-effective) version of the above result was obtained by Larsen and Pink [43] without relying on the classification of finite simple groups. It is clear that  $H$  is a soluble by  $Lie^*(p)$  subgroup.

*Remark 89.* Let  $P$  be a perfect  $p$ -generated subgroup of  $GL(n, \mathbb{F}_p)$ . Using Proposition 88 and [28, Lemma 3] one can easily show that every element of  $P$  is the product of  $g(n)$  commutators where  $g(n)$  depends on  $n$ . This could be used to replace the (rather more difficult) Proposition 78 in the proof of Theorem 85.

The rest of this section will be devoted to proving results concerning subsets  $\alpha$  of  $GL(n, \mathbb{F}_p)$  that satisfy  $|\alpha^3| \leq K|\alpha|$ . We consider the group  $G = \langle \alpha \rangle$  and we will establish step by step a close relationship between  $\alpha$  (and its powers) and the structure of  $G$  described in Proposition 88. Throughout the proof we need to establish several auxiliary results.

**Proposition 90.** *Let  $G$  be a group and  $\alpha \subseteq G$  a symmetric generating set with  $1 \in \alpha$ . If  $H$  is a normal subgroup of index  $t$  in  $G$  then  $\alpha^{2t} \cap H$  generates  $H$ .*

*Proof.* It is clear that  $\alpha^{t-1}$  contains a full system of coset representatives  $g_1, \dots, g_t$  of  $G/H$ . It is well-known (see [58, Theorem 2.6.9]) that  $H$  is generated by elements of the form  $g_i a g_j^{-1}$  where  $a \in \alpha$ .  $\square$

**Proposition 91.** *Let  $\alpha$  be a finite subset of a group  $G$  and  $\tilde{G} = G/N$  a quotient of  $G$ . Set  $\tilde{\alpha} = \alpha N/N$ . Then  $|\alpha^4|/|\alpha| \geq |\tilde{\alpha}^3|/|\tilde{\alpha}|$ . Moreover, if  $\alpha$  is symmetric and  $1 \in \alpha$  then  $(|\tilde{\alpha}^3|/|\tilde{\alpha}|)^2 \geq |\tilde{\alpha}^3|/|\tilde{\alpha}|$ .*

*Proof.* There is a coset  $gN$  of  $N$  such that  $|\alpha \cap gN| \geq |\alpha|/|\tilde{\alpha}|$ . We may assume that  $g \in \alpha$ . Let  $\{g_i\}$  be a system of representatives of the cosets in  $\tilde{\alpha}^3$  with  $g_i \in \alpha^3$ . Then the sets  $g_i(\alpha \cap gN)$  are disjoint subsets of  $\alpha^4$  hence  $|\alpha^4| \geq |\tilde{\alpha}^3| |\alpha|/|\tilde{\alpha}|$  as required. The other inequality follows then from Proposition 74.(b).  $\square$

**Proposition 92.** *Let  $H$  be a soluble by  $\text{Lie}^*(p)$  subgroup of  $GL(n, \mathbb{F}_p)$  and  $\gamma \leq H$  a symmetric generating set with  $1 \in \gamma$ . Assume that  $\gamma$  satisfies  $|\gamma^3| \leq K|\gamma|$  for some  $K > 2$ . Then there is a soluble by  $\text{Lie}^*(p)$  normal subgroup  $S$  of  $H$  such that  $\gamma^6 \cap S$  projects onto all Lie type simple quotients of  $S$  and  $\gamma$  is covered by  $K^c$  cosets of  $S$ , where  $c = c(n)$  depends only on  $n$ .*

*Proof.* Let  $H/N \cong L$  be a Lie type simple quotient of  $H$  and set  $\tilde{\gamma} = \gamma N/N$ . The Lie rank of  $L$  is at most  $n$  (see [21] and [38, Proposition 5.2.12]). Now  $|\tilde{\gamma}^3| \leq K^2|\tilde{\gamma}|$  by Proposition 91. Hence by Theorem 77 we have two possibilities; either  $|\tilde{\gamma}| \geq |\tilde{\gamma}^3|/K^2 = |L|/K^2$  or  $|\tilde{\gamma}| \leq K^b$  where  $b = b(n)$  depends only on  $n$ . Set  $c = 6n^2(2 + nb)$ . If  $(p-1)/2 \leq K^{3(2+nb)}$  then we have  $|GL(n, \mathbb{F}_p)| < K^c$  (since  $K > 2$ ) and our statement holds for  $S = 1$ .

Otherwise let  $H/N_j \cong L_j$  ( $j = 1, \dots, t$ ) be all the Lie type simple quotients of  $H$  (there are at most  $n$  such quotients e.g. by [44, Corollary 3.3]). Let  $H/N_1, H/N_2, \dots, H/N_i$  be the quotients for which the second possibility holds. Consider the subgroup  $S = N_1 \cap \dots \cap N_i$ . It is clear that  $S$  is a soluble by  $\text{Lie}^*(p)$  normal subgroup and its Lie type simple quotients are  $S/(S \cap N_{i+1}), \dots, S/(S \cap N_t)$ . Moreover  $\gamma$  is covered by at most  $K^{nb}$  cosets of  $S$ .

It remains to prove that  $\gamma^6 \cap S$  projects onto, say,  $S/(S \cap N_{i+1})$ . Consider the quotient group  $\overline{H} = H/(S \cap N_{i+1})$ . The image  $\overline{\gamma}$  of  $\gamma$  in  $\overline{H}$  is covered by at most  $K^{nb}$  cosets of  $\overline{S} = S/(S \cap N_{i+1}) \cong L_{i+1}$  and we have  $|\overline{\gamma}| \geq |\overline{S}|/K^2$ . This implies that some coset of  $\overline{S}$  in  $\overline{H}$  contains at least  $|\overline{S}|/K^{2+nb}$  elements of  $\overline{\gamma}$  and it follows that  $|\overline{\gamma}^2 \cap \overline{S}| \geq |\overline{S}|/K^{2+nb}$ . By Remark 76 the minimal degree of a complex representation of  $\overline{S}$  is at least  $(p-1)/2 > (K^{2+nb})^3$  hence by Proposition 75 we have  $(\overline{\gamma}^2 \cap \overline{S})^3 = \overline{S}$ , which implies our statement.  $\square$

**Proposition 93.** *Assume that a symmetric subset  $\alpha$  of a group  $G$  is covered by  $x$  right cosets of a subgroup  $H$  and  $\alpha^2 \cap H$  is covered by  $y$  right cosets of a subgroup  $S \leq H$ . Then  $\alpha$  is covered by  $xy$  right cosets of  $S$ .*

*Proof.* We have  $\alpha \subseteq Hg_1 \cup \dots \cup Hg_x$  and  $\alpha^2 \cap H \subseteq Sh_1 \cup \dots \cup Sh_y$  where the coset representatives  $g_i$  are chosen from  $\alpha$ . If  $a \in \alpha \cap Hg_i$  then by our assumptions  $ag_i^{-1} \in Sh_j$  for some  $j$ , hence  $a \in Sh_j g_i$ . Therefore  $\alpha \subseteq \bigcup_i \bigcup_j Sh_j g_i$ .  $\square$

**Proposition 94.** *Let  $G$  and  $H$  be as in Proposition 88. Let  $\alpha$  be a symmetric set of generators of  $G$  with  $1 \in \alpha$  satisfying  $|\alpha^3| \leq K|\alpha|$  for some  $K > 2$ . Set  $\gamma = \alpha^{2f(n)} \cap H$ .*

- a) *The set  $\gamma$  generates  $H$  and satisfies  $|\gamma^3| \leq K_0|\gamma|$  where  $K_0 = K^{7f(n)}$ .*
- b) *Let  $S$  be the subgroup constructed from  $\gamma$  and  $H$  in the proof of Proposition 92. If  $p \geq K_0^{b_0(n)}$  (where  $b_0(n) = b(n) + 4$  with the same  $b(n)$  as in the proof of Proposition 92) then  $S$  is normal in  $G$ .*
- c)  *$\alpha$  is covered by at most  $K_0^{c_0(n)}$  cosets of  $S$  (where  $c_0(n) = c(n) + \log(f(n))/\log(2)$  with the same  $c(n)$  as in Proposition 92).*
- d) *The commutator subgroup  $S'$  is an extension of a  $p$ -group by a  $\text{Lie}^{**}(p)$ -group.*

*Proof.* Consider  $\beta = \alpha^{f(n)}$ . By Proposition 90  $\gamma = \beta^2 \cap H$  generates  $H$ . Using Lemma 50 and Proposition 74 we see that

$$\frac{|\gamma^3|}{|\gamma|} \leq \frac{|\beta^6 \cap H|}{|\beta^2 \cap H|} \leq \frac{|\beta^7|}{|\beta|} \leq \frac{|\alpha^{7f(n)}|}{|\alpha|} \leq K^{7f(n)}$$

which proves (a). Part (c) follows using Proposition 93. Part (d) follows from Proposition 88.

It remains to prove (b). If  $H/N_j$  are all the Lie type simple quotients of  $H$  then  $N = \bigcap_j N_j$  is the soluble radical of  $H$ . Consider the quotient  $\overline{G} = G/N$ . The set  $\overline{\gamma}$  generates the normal subgroup  $\overline{H} \triangleleft \overline{G}$ . For each  $a \in \alpha$  the conjugation by  $\overline{a} \in \overline{\alpha}$  is an automorphism of  $\overline{H}$ . Now  $\overline{H}$  is the direct product of nonabelian simple groups and an automorphism of  $\overline{H}$  permutes these factors (because the direct decomposition is unique).

If  $S$  is not normal in  $G$  then there is a Lie type simple quotient of  $H$ , say  $H/N_1 \cong L_1$  and an element  $a \in \alpha$  such that  $\gamma$  projects onto at most  $K_0^{b(n)}$  elements of  $H/N_1$  and  $a^{-1}\gamma a$  projects onto at least  $|L_1|/K_0^2$  elements of  $H/N_1$ . Note that  $a^{-1}\gamma a = a^{-1}(\beta^2 \cap H)a \subseteq \beta^4 \cap H$ . By the



above we have  $|\beta^2 \cap H| = |\gamma| \leq |\gamma^2 \cap N_1| K_0^{b(n)}$ . On the other hand,  
 $|\beta^8 \cap H| \geq |(\beta^4 \cap H)(\beta^2 \cap H)^2| \geq |(a^{-1}\gamma a)(\gamma^2 \cap N_1)| \geq \frac{|L_1|}{K_0^2} |\gamma^2 \cap N_1|$ .  
 Therefore  $\frac{|\beta^8 \cap H|}{|\beta^2 \cap H|} \geq |L_1|/K_0^{2+b(n)}$ . But we have  $\frac{|\beta^8 \cap H|}{|\beta^2 \cap H|} \leq \frac{|\beta^9|}{|\beta|} \leq K^{9f(n)} < K_0^2$ . We obtain that  $|L_1| < K_0^{4+b(n)}$ , a contradiction.  $\square$

As we saw above, a subset  $\alpha$  of  $GL(n, \mathbb{F}_p)$  with  $|\alpha^3| \leq K|\alpha|$  is essentially contained in a normal subgroup  $S$  of  $G = \langle \alpha \rangle$  such that a small power of  $\alpha$  projects onto all Lie type simple quotients of  $S$ . We proceed to show that the latter property also holds for the last term  $P$  of the derived series of  $S$ . Later we will prove that a small power of  $\alpha$  in fact generates  $P$  (see Proposition 100).

**Proposition 95.** *Let  $S$  be a soluble by  $Lie^*(p)$  subgroup of  $GL(n, \mathbb{F}_p)$ . Let  $1 \in \alpha$  be a symmetric subset of  $S$  which projects onto all Lie type simple quotients of  $S$ . Let  $P$  be the last term of the derived series of  $S$ . Then  $P$  is a perfect soluble by  $Lie^*(p)$  subgroup and  $\alpha^c \cap P$  projects onto all Lie type simple quotients of  $P$  where  $c = c(n)$  depends only on  $n$ .*

*Proof.* Let  $S/N_i$  be the Lie type simple quotients of  $S$ . The commutator subgroup  $S'$  is clearly also a soluble by  $Lie^*(p)$  subgroup and its Lie type simple quotients are the  $S'/(S' \cap N_i) \cong S/N_i$ . We need the following.

**Claim 96.**  *$S' \cap \alpha^b$  projects onto  $S'/(S' \cap N_i)$  for all  $i$  where  $b = b(n)$  depends only on  $n$ .*

To see this fix  $i$  and consider the quotient  $\overline{S} = S/(S' \cap N_i)$ . This quotient is the direct product of  $S'/(S' \cap N_i)$  and  $N_i/(S' \cap N_i) \cong S/S'$  (since these have no common quotients). Take two elements  $a, b \in \alpha$  which project onto noncommuting elements of  $S/N_i$ . The image of the commutator  $[a, b] \in \alpha^4$  in  $\overline{S}$  is a nontrivial element of  $S'/(S' \cap N_i)$ . Each element of  $S' \cap N_i$  appears as the first coordinate of some element of the image  $\overline{\alpha}$  of  $\alpha$  in  $\overline{S}$ . Taking conjugates of  $\overline{[a, b]}$  with these elements we obtain that the whole conjugacy class of  $[a, b]$  in the simple group  $S'/(S' \cap N_i)$ . But this group has Lie rank at most  $n$  and therefore each element of  $S'/(S' \cap N_i)$  is the product of at most  $a(n)$  conjugates of an arbitrary nontrivial element where  $a(n)$  depends only on  $n$  (in fact  $a(n)$  is a linear function of  $n$  by [42]). Therefore  $\alpha^{6a(n)} \cap S'$  projects onto  $S'/(S' \cap N_i)$  as claimed.

The length of the derived series of any subgroup of  $GL(n, \mathbb{F}_p)$  is bounded in  $n$  (in fact there is a logarithmic bound). Hence our statement follows from the Claim by an obvious induction argument.  $\square$

**Definition 97.** If  $L = L_1 \times \cdots \times L_k$  is a direct product of isomorphic groups,  $D$  a subgroup of  $L$  isomorphic to  $L_1$  which projects onto each direct factor then we call  $D$  a *diagonal subgroup*.

**Proposition 98.** *Let  $L = L_1 \times \cdots \times L_k$  be a direct product of  $k$  non-abelian simple groups and  $T$  a subgroup which projects onto all simple quotients of  $L$ . Then any chain of subgroups between  $T$  and  $L$  has length at most  $k$ .*

*Proof.* Let  $H$  be a subgroup of  $L$  which projects onto all simple quotients of  $L$  (i.e. a subdirect product). Then there is a partition of the set of simple groups  $L_i$  such that the groups in any partition-class are isomorphic and  $H$  is the direct product of diagonal subgroups corresponding to these partition-classes (see [5, Proposition 3.3]). Our statement follows.  $\square$

**Proposition 99.** *Let  $L$  be a  $\text{Lie}^{**}(p)$ -group and  $T$  a subgroup which projects onto  $L/\mathcal{Z}(L)$ . Then  $T = L$ .*

*Proof.* We have  $T\mathcal{Z}(L) = L$  which implies that  $T$  is a normal subgroup of  $L$ . Moreover,  $L/T$  is abelian and since  $L$  is perfect, we have  $T = L$ .  $\square$

**Proposition 100.** *Let  $H$  be a subgroup of  $GL(n, \mathbb{F}_p)$ ,  $S$  a soluble by  $\text{Lie}^*(p)$  normal subgroup of  $H$  and  $P$  the last term in the derived series of  $S$ . Assume that  $P$  is an extension of a  $p$ -group by a  $\text{Lie}^{**}(p)$ -group. Let  $1 \in \gamma$  be a symmetric generating set of  $H$ . Assume that  $\gamma^t \cap P$  projects onto all Lie type simple quotients of  $P$  for some integer  $t$ . Then  $\gamma^{t+2n+2n^2} \cap P$  generates  $P$ .*

*Proof.* Set  $Q_i = \langle \gamma^i \cap P \rangle$ . We first show that  $Q_{t+2n}$  projects onto  $P/O_p(P)$ . Since  $P/O_p(P)$  is a  $\text{Lie}^{**}(p)$ -group, by Proposition 99 it is sufficient to prove that  $Q_{t+2n}$  projects onto the central quotient of  $P/O_p(P)$ , which is exactly  $P/\text{Sol}(P)$ . Denote  $P/\text{Sol}(P)$  by  $\overline{P}$  and let  $\overline{Q}_i$  denote the image of  $Q_i$  in  $\overline{P}$ . We need the following.

**Claim 101.** *If  $i \geq t$  and  $\overline{Q}_i \neq \overline{P}$  then  $|Q_{i+2}|$  is strictly greater than  $|Q_i|$ .*

To see this, observe that  $\overline{Q}_i$  projects onto all simple quotients of  $\overline{P}$  and the only normal subgroup of  $\overline{P}$  with this property is  $\overline{P}$  itself. By our assumptions there is an  $a \in \gamma$  for which  $\overline{Q}_i$  and its conjugate  $\overline{Q}_i^a$  are different subgroups of  $\overline{Q}_{i+2}$ . This implies the claim.

As noted earlier,  $\overline{P}$  is the direct product of at most  $n$  simple groups. Hence by Proposition 98 any chain of subgroups containing  $\overline{Q}_t$  has length at most  $n$ . By the above claim  $Q_{t+2n}$  projects onto  $P/\text{Sol}(P)$ , hence onto  $P/O_p(P)$  as stated. We also need the following.

**Claim 102.** *If  $Q_i$  is not a normal subgroup of  $H$  and  $i \geq t + 2n$  then  $|Q_{i+2}| \geq |Q_i| \cdot p$ .*

To see this, consider as above an element  $a \in \gamma$  which does not normalise  $Q_i$ . Then  $Q_i$  and  $Q_i^a$  are different subgroups of  $P$  generated by subsets of  $\gamma^{i+2}$ . Hence  $P \geq Q_{i+2} \not\leq Q_i$ . By our assumptions  $|P : Q_i|$  is a power of  $p$  which implies the Claim.

Repeated applications of the Claim yield an ascending chain of subgroups  $Q_{t+2n} \leq Q_{t+2n+2} \leq Q_{t+2n+4} \leq \cdots \leq Q_{t+2n+2k} = Q \leq P$  which of course has length less than  $n^2$ . The last term  $Q$  of this chain is normal in  $H$  hence in  $S$ . By our assumptions all nonabelian simple composition factors of  $S$  are among the composition factors of  $Q$  (with multiplicities). Therefore  $S/Q$  must be soluble i.e.  $Q = P$ .  $\square$

**Proposition 103.** *Let  $G$  be a finite group and  $\alpha$  a generating set such that  $\alpha^k$  contains the subgroup  $P$ . Then*

$$\frac{\max_{g \in G} |\alpha \cap gP|}{|P|} \geq \frac{|\alpha|}{|\alpha^{k+1}|}.$$

*Proof.* Let  $t$  be the number of cosets of  $P$  which contain elements of  $\alpha$ . Then we have  $\max_g |\alpha \cap gP| \cdot t \geq |\alpha|$ . On the other hand it is clear that  $|\alpha^{k+1}| \geq t|P|$ . Hence

$$\frac{|\alpha^{k+1}|}{|P|} \geq t \geq \frac{|\alpha|}{\max_{g \in G} |\alpha \cap gP|}$$

as required.  $\square$

Now we are ready to prove our main results concerning subsets  $\alpha$  of  $GL(n, \mathbb{F}_p)$  with  $|\alpha^3| \leq K|\alpha|$ .

**Theorem 104.** *Let  $\alpha$  be a symmetric subset of  $GL(n, \mathbb{F}_p)$  satisfying  $|\alpha^3| \leq K|\alpha|$  for some  $K \geq 1$ . Then  $GL(n, \mathbb{F}_p)$  has two subgroups  $S \geq P$ , both normalised by  $\alpha$ , such that  $P$  is perfect,  $S/P$  is soluble, a coset of  $P$  contains at least  $|P|/K^{c(n)}$  elements of  $\alpha$  and  $\alpha$  is covered by  $K^{c(n)}$  cosets of  $S$  where  $c(n)$  depends on  $n$ .*

*Proof.* If  $K \leq 2$  then let  $S$  be the subgroup generated by  $\alpha$  and  $P$  the last term of the derived series of  $S$ . By Proposition 73 we have  $\alpha^3 = S$  hence  $|\alpha| \geq |S|/K$ , which implies that some coset of  $P$  contains at least  $|P|/K$  elements. If  $K > 2$  and  $p < K^{7f(n)b_0(n)}$  (with the notation of Proposition 94) then we set  $S = P = \{1\}$ . Now we have  $|\alpha| < K^{7f(n)b_0(n)n^2}$  which proves our statement in this case. From now on we assume that  $K > 2$  and  $p \geq K^{7f(n)b_0(n)}$ .

Let  $S$  be as in Proposition 94. Then  $\alpha$  is covered by  $K^{7f(n)c_0(n)}$  cosets of  $S$ . By Proposition 92 the set  $\alpha^{12f(n)} \cap S$  projects onto all Lie type simple quotients of  $S$ .

Let  $P$  be the last term of the derived series of  $S$ . Proposition 94.(d) implies that  $P$  is an extension of a  $p$ -group by a  $Lie^{**}(p)$ -group, in particular  $P$  is a  $p$ -generated group. Let  $c_1(n)$  be the constant of Proposition 95 (denoted there by  $c(n)$ ), set  $c_2(n) = 2f(n)(6c(n) + 2n + 2n^2)$ .  $\alpha^{c_2(n)} \cap P$  generates  $P$  and projects onto all Lie type simple quotients of  $P$  by Proposition 95 and Proposition 100. By Theorem 85 if  $c(n) \geq c_2(n)d(n)$  then  $\alpha^{c(n)}$  contains  $P$ .

Using Proposition 103 and Proposition 74.(b) we obtain that some coset of  $P$  contains at least

$$\frac{|P||\alpha|}{|\alpha^{c(n)+1}|} \geq \frac{|P|}{K^{c(n)}}$$

elements of  $\alpha$ . The proof is complete.  $\square$

The following is a slightly stronger version of Theorem 7.

**Corollary 105.** *Let  $\alpha$  be a symmetric subset of  $GL(n, \mathbb{F}_p)$  satisfying  $|\alpha^3| \leq K|\alpha|$  for some  $K \geq 1$ . Then  $GL(n, \mathbb{F}_p)$  has two subgroups  $S \geq P$ , both normalised by  $\alpha$ , such that  $P$  is perfect,  $S/P$  is soluble, a coset of  $P$  is contained in  $\alpha^3$  and  $\alpha$  is covered by  $K^{c(n)}$  cosets of  $S$  where  $c(n)$  depends on  $n$ .*

*Proof.* If  $K \leq 2$  then  $\alpha^3 = \langle \alpha \rangle$  by Proposition 73 and our statement follows. Let  $c'(n)$  the constant in Theorem 104. If  $\frac{p-1}{2} \leq K^{3c'(n)}$  and  $K > 2$  then it follows that  $|\alpha| \leq K^{6c'(n)n^2}$  hence our statement holds for  $S = P = 1$  with  $c(n) = 6c'(n)n^2$ .

We assume that  $K > 2$  and  $K^{3c'(n)} < \frac{p-1}{2}$ . Let  $S$  and  $P$  be as in Theorem 104. By that theorem there is a subset  $X$  of  $P$  of size at least  $|P|/K^{c'(n)}$  such that  $aX \subseteq \alpha$  for some  $a \in \alpha$ . Now

$$\alpha^3 \supseteq aXaXaX = a^3(a^{-2}Xa^2)(a^{-1}Xa)X.$$

By our assumptions and Proposition 79.(b) if  $k$  is the minimal degree of a complex representation of  $P$  then we have  $|a^{-2}Xa^2||a^{-1}Xa||X| \geq |P|^3/k$ . hence by Proposition 75 we have  $\alpha^3 \supset a^3P$  as required.  $\square$

To obtain a characterisation for symmetric subsets  $\alpha$  of  $GL(n, \mathbb{F}_p)$  satisfying  $|\alpha^3| \leq K|\alpha|$  with polynomially bounded constants (as in Theorem 104) seems to be a very difficult task. As another step towards such a characterisation we mention the following (folklore) conjecture.

**Conjecture 106.** *Let  $1 \in \alpha$  be a symmetric subset of  $GL(n, \mathbb{F}_p)$  satisfying  $|\alpha^3| \leq K|\alpha|$  for some  $K \geq 1$ . Then  $GL(n, \mathbb{F}_p)$  has two subgroups  $S \triangleright P$  such that  $S/P$  is nilpotent,  $P$  is contained in  $\alpha^{c(n)}$  and  $\alpha$  is covered by  $K^{c(n)}$  cosets of  $S$  where  $c(n)$  depends on  $n$ .*

The following is well-known.

**Proposition 107.** *Let  $S$  be a finite group and  $P$  a normal subgroup with  $S/P$  soluble. If  $C$  is a minimal subgroup such that  $PC = S$  then  $C$  is soluble.*

*Proof.* Let  $M$  be a maximal subgroup of  $C$ . If  $M$  does not contain  $C \cap P$  then  $(C \cap P)M = C$  which implies  $PM = PC = S$ , a contradiction. Hence all maximal subgroups of  $C$ , and therefore its Frattini subgroup  $\Phi(C)$  contain  $C \cap P$ . But  $\Phi(C)$  is nilpotent, hence  $C$  is soluble.  $\square$

Theorem 104 and Proposition 107 can be used to show that if Conjecture 106 holds in the case when  $\langle \alpha \rangle$  is soluble then it holds in general. We omit the details. <sup>4</sup>

### 13. LINEAR GROUPS OVER ARBITRARY FIELDS

In this section we develop another method to show that a certain spreading system  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading. As in the proof of Theorem 70, we find an appropriate CCC-subgroup  $A < G$ , but now we study the case when  $A$  has infinitely many  $\langle \alpha \rangle$ -conjugates.

We use the resulting new spreading theorem (Theorem 108) inductively to show that if  $\alpha$  is a non-growing subset of  $GL(n, \mathbb{F})$ ,  $\mathbb{F}$  an arbitrary field, then  $\langle \alpha \rangle$  is essentially contained in a virtually soluble group (see Corollary 111).

Combining Corollary 111 with various results on finite groups (in particular Theorem 4) we obtain Theorem 10, our main result on arbitrary finitely generated linear groups.

**Theorem 108.** *For all parameters  $N > 0$ ,  $\Delta > 0$  and  $\frac{1}{119N^3} > \varepsilon > 0$  there is an integer  $M = M_\infty(N, \varepsilon) > 0$  and a real  $K = K_\infty(N, \Delta, \varepsilon) > 0$  with the following property.*

*Let  $\alpha|G$  be an  $(N, \Delta, K)$ -bounded spreading system. Then either  $\langle \alpha \rangle \cap G$  is virtually nilpotent or  $\alpha|G$  is  $(\varepsilon, M, K)$ -spreading. Moreover, our construction of the subgroup of spreading is uniquely determined.*

*Proof.* Using the bounds from Lemma 65 and Lemma 63 we set

$$M_{\text{CCC}} = M_{\text{CCC}}(N, \varepsilon) \ , \quad M_s = M_s(N, \varepsilon) \ ,$$

---

<sup>4</sup>Very recently Gill and Helfgott [25] have proved Conjecture 106 in the soluble case.

$$M = M_{\text{CCC}} \cdot M_s ,$$

$$K = \max \left( \Delta, K_{\text{CCC}}(N, \Delta, \varepsilon), K_s(N, \Delta, \varepsilon) \right) .$$

Suppose that  $\alpha|G$  is not  $(\varepsilon, M, K)$ -spreading. In particular, it is not  $(N\varepsilon, M_{\text{CCC}}, K)$ -spreading either, hence

$$\mu(\alpha^{M_{\text{CCC}}}, G) < (1 + N\varepsilon)\mu(\alpha, G) .$$

If  $G$  is nilpotent then there is nothing to prove, so we assume that  $G$  is non-nilpotent. Using Lemma 65 we obtain a CCC-subgroup  $A \subseteq G$  containing a single maximal torus  $T$  such that

$$\begin{aligned} \mu(\alpha^{M_{\text{CCC}}}, A) &> (1 - \varepsilon \cdot 16N) \mu(\alpha, G) > \\ &> \left(1 - \frac{1}{7N^2}\right) (1 + N\varepsilon) \mu(\alpha, G) > \left(1 - \frac{1}{7N^2}\right) \mu(\alpha^{M_{\text{CCC}}}, G) . \end{aligned}$$

In particular  $A$  is not normal in  $G$ . If  $A$  has infinitely many  $\langle \alpha \rangle$ -conjugates then  $\alpha^{M_{\text{CCC}}}|G$  is  $(\varepsilon, M_s, K)$ -spreading by Lemma 63, a contradiction. So  $A$  has finitely many  $\langle \alpha \rangle$ -conjugates. Then  $T$  has finitely many  $\langle \alpha \rangle$ -conjugates, hence  $\langle \alpha \rangle \cap \mathcal{N}_G(T)$  has finite index in  $\langle \alpha \rangle \cap G$ .

On the other hand  $\mathcal{N}_G(T) = \mathcal{N}_G(\mathcal{C}_G(T))$ , and  $\mathcal{C}_G(T)$  is a Cartan subgroup, so it is nilpotent and has finite index in its normaliser. Therefore  $\mathcal{N}_G(T)$  is virtually nilpotent, hence  $\langle \alpha \rangle \cap G$  is also virtually nilpotent.  $\square$

Our plan is to apply Theorem 108, then apply it to the subgroup of spreading, then apply it again to the new subgroup of spreading, and so on, until we eventually arrive to a subgroup whose intersection with  $\langle \alpha \rangle$  is virtually nilpotent.

We need the following fact:

**Proposition 109** (Freiman [22]). *Let  $\alpha$  be a finite subset of a group  $G$ . If  $|\alpha \cdot \alpha| < \frac{3}{2}|\alpha|$ , then  $S := \alpha \cdot \alpha^{-1}$  is a finite group of order  $|\alpha \cdot \alpha|$ , and  $\alpha \subset S \cdot x = x \cdot S$  for some  $x$  in the normaliser of  $S$ .*

**Proposition 110.** *For all parameters  $n > 0$ ,  $d > 0$  there are integers  $m = m_{\text{nilp}}(n, d) > 0$  and  $D = D_{\text{nilp}}(n, d) > 0$  with the following property.*

*Let  $G \leq GL(n, \overline{\mathbb{F}})$  be a (possibly non-connected) closed subgroup and  $\alpha \leq G$  a finite subset such that  $\dim(G) \geq 1$ ,  $\deg(G) \leq d$  and  $|\alpha^3| \leq K|\alpha|$  for some  $K$ . Then either  $|\alpha| \leq K^m$  or one can find a connected closed subgroup  $H \leq G$  normalised by  $\alpha$  such that  $\dim(H) \geq 1$ ,  $\deg(H) \leq D$  and  $\langle \alpha \rangle \cap H$  is virtually nilpotent.*

*Proof.* During the proof we encounter several lower bounds for  $m$ , we assume that our  $m$  satisfies them all. Similarly, we shall establish several alternative upper bounds on  $\deg(H)$ , we set  $D$  to be the maximum

of these bounds. If  $\mathcal{K} < \frac{3}{2}$  then  $\langle \alpha \rangle$  is virtually cyclic by Proposition 109 and the lemma holds with  $H = G^0$ . If  $\mathcal{C}_G(\alpha)$  is infinite then we take  $H = \mathcal{C}_G(\alpha)^0$  (see Fact 36). So we assume that  $\mathcal{K} \geq \frac{3}{2}$ ,  $\mathcal{C}_G(\alpha)$  is finite and  $|\alpha| > \mathcal{K}^m$ . By Proposition 74.(a) we can assume that  $\alpha$  is symmetric and  $1 \in \alpha$ . We order the set  $\alpha$ .

By assumption  $|G : G^0| \leq d$ , hence  $|\alpha^2 \cap G^0| \geq \frac{|\alpha|}{d}$ . We set  $\varepsilon = \frac{1}{120n^6}$ ,  $G_0 = G^0$ , and construct by induction a sequence of length at most  $n^2$  of connected closed subgroups  $G_0 > G_1 > G_2 > \dots$  normalised by  $\alpha$  and corresponding constants  $e_i, K_i$  such that

$$\dim(G_i) \geq 1, \quad \deg(G_i) \leq K_i, \quad |\alpha^{e_i} \cap G_i| \geq \left(\frac{|\alpha|}{d}\right)^{\dim(G_i)/n^2}.$$

It will be clear from the construction that all of the appearing constants (i.e.  $e_i, K_i, \Delta_i$  and  $M$ , see below) depend only on  $n$  and  $d$ . We already defined  $G_0$ , our statement holds with  $K_0 = d$  and  $e_0 = 2$  (since closed subgroups of  $GL(n, \overline{\mathbb{F}})$  have dimension at most  $n^2$ ). Suppose that  $G_i, K_i$  and  $e_i$  are already constructed for some  $i \geq 0$ . We assume that  $\langle \alpha \rangle \cap G_i$  is not virtually nilpotent, since otherwise the lemma holds with  $H = G_i$  (whose degree is bounded in terms of  $n$  and  $d$ ). According to Proposition 31 the numerical invariants  $\deg(G_i)$ ,  $\text{mult}(G_i)$  and  $\text{inv}(G_i)$  are bounded from above by a certain constant  $\Delta_i = \Delta_i(n^2, K_i)$ . Recall from Theorem 108 the constants  $M = M_\infty(n^2, \varepsilon)$  and  $K_{i+1} = K_\infty(n^2, \Delta_i, \varepsilon)$ . We assume that  $m$  is large enough so that  $\mathcal{K}^m > \left(\frac{3}{2}\right)^m > d(K_{i+1})^{n^2}$ . Then the  $\alpha^e |G_i$  are  $(n^2, \Delta_i, K_{i+1})$ -bounded spreading systems for all  $e \geq e_i$ , hence according to Theorem 108 they are  $(\varepsilon, M, K_{i+1})$ -spreading.

Let us consider the spreading systems  $\alpha^{e_i M^j} |G_i$  for  $j = 0, 1, 2, \dots, J-1$ , where  $J = 2^{\frac{n^2}{\varepsilon}} = 240n^8$ . Suppose now that for each  $i$ ,  $G_i$  itself is the subgroup of spreading obtained above using Theorem 108. Then  $\mu(\alpha^{e_i M^J}, G_i) \geq (1 + \varepsilon)^J \mu(\alpha, G_i)$  i.e.

$$|\alpha^{e_i M^J} \cap G_i| \geq |\alpha^{e_i} \cap G_i|^{(1+\varepsilon)^J} > \left(\frac{|\alpha|}{d}\right)^{J\varepsilon/n^2} = \left(\frac{|\alpha|}{d}\right)^2 \geq |\alpha| \frac{\mathcal{K}^m}{d^2}.$$

On the other hand, by Proposition 74.(b) we have  $|\alpha^{e_i M^J}| \leq |\alpha| \mathcal{K}^{e_i M^J - 2}$ . We rule this case out by choosing  $m \geq e_i M^J + \frac{\log(d^2)}{\log(3/2)}$ . Then there is a value  $j_0 < J$  such that the corresponding subgroup of spreading is a proper subgroup of  $G_i$ . This subgroup will be our  $G_{i+1}$ , and we set  $e_{i+1} = e_i M^{j_0}$ . We obtain

$$|\alpha^{e_{i+1}} \cap G_{i+1}| \geq |\alpha^{e_i M^{j_0+1}} \cap G_{i+1}| \geq |\alpha^{e_i M^{j_0}} \cap G_i|^{\frac{\dim(G_{i+1})}{\dim(G_i)}} \geq$$

$$\geq \left| \alpha^{e_i} \cap G_i \right|^{\frac{\dim(G_{i+1})}{\dim(G_i)}} \geq \left( \frac{|\alpha|}{d} \right)^{\frac{\dim(G_i)}{n^2} \frac{\dim(G_{i+1})}{\dim(G_i)}} \geq \left( \frac{|\alpha|}{d} \right)^{\frac{\dim(G_{i+1})}{n^2}},$$

the induction step is complete. The dimensions  $\dim(G_i)$  strictly decrease as  $i$  grows, hence the induction must stop in at most  $n^2$  steps. But the only way it can stop is to produce the required subgroup  $H$ .  $\square$

Iterating the previous lemma we obtain that a non-growing subset  $\alpha \subset GL(n, \overline{\mathbb{F}})$  is covered by a few cosets of a virtually soluble group. In the proof we need an auxiliary subgroup  $G$  in order to do induction on  $\dim(G)$ . For applications the only interesting case is  $G = GL(n, \overline{\mathbb{F}})$ ,  $\deg(G) = 1$ .

**Corollary 111.** *Let  $G \leq GL(n, \overline{\mathbb{F}})$  be a (possibly non-connected) closed subgroup and  $\alpha \subseteq G$  a finite subset. Suppose that  $|\alpha^3| \leq \mathcal{K}|\alpha|$  for some  $\mathcal{K}$ . Then there is a virtually soluble normal subgroup  $\Delta \triangleleft \langle \alpha \rangle$  and a bound  $m = m(n, \deg(G))$  such that the subset  $\alpha$  can be covered by  $\mathcal{K}^m$  cosets of  $\Delta$ .*

*Proof.* During the proof we encounter several lower bounds for  $m$ , we assume that our  $m$  satisfies them all. We prove the corollary by induction on  $N = \dim(G)$ . If  $\mathcal{K} < \frac{3}{2}$  then  $\langle \alpha \rangle$  is virtually cyclic by Proposition 109 and the lemma holds with  $\Delta = \langle \alpha \rangle$ . If  $|\alpha| \leq \mathcal{K}^m$  then our statement holds with  $\Delta = \{1\}$ . So we assume that  $\mathcal{K} \geq \frac{3}{2}$  and  $|\alpha| > \mathcal{K}^m$ . If  $\dim(G) = 0$  then  $|\alpha| \leq \deg(G)$ , we exclude this case by choosing  $m$  large enough.

Suppose that  $m \geq m_{\text{nilp}}(n, \deg(G))$ . Applying Proposition 110 we obtain a subgroup  $H$  normalised by  $\alpha$  such that  $\langle \alpha \rangle \cap H$  is virtually nilpotent,  $\dim(H) \geq 1$ , and  $\deg(H)$  is bounded in terms of  $n$  and  $\deg(G)$ . Consider the algebraic group  $\overline{G} = \mathcal{N}_G(H)/H$ , let  $\overline{\alpha} \subseteq \overline{G}$  denote the image of  $\alpha$ . By Proposition 91 we have  $|\overline{\alpha}^3| \leq \mathcal{K}^2|\overline{\alpha}|$ . By Proposition 80 and Fact 20.(f) there is an embedding  $\overline{G} \leq GL(n', \overline{\mathbb{F}})$  where  $n'$  and  $\deg(\overline{G})$  are bounded in terms of  $n$ ,  $\deg(G)$  and  $\deg(H)$ . Clearly  $\dim(\overline{G}) < \dim(G)$ , so by the induction hypothesis we obtain a virtually soluble normal subgroup  $\overline{\Delta} \triangleleft \langle \overline{\alpha} \rangle$  such that  $\overline{\alpha}$  is covered by  $\mathcal{K}^{2m(n', \deg(\overline{G}))}$  cosets of  $\overline{\Delta}$ . We define  $\Delta$  to be the preimage of  $\overline{\Delta}$  in  $\langle \alpha \rangle$ . Then  $\Delta$  is virtually soluble since the class of virtually soluble groups is closed under extensions (see e.g. [37]). The induction step is complete.  $\square$

The following consequence of well-known results is of independent interest.

**Lemma 112.** *Let  $\Delta$  be a virtually soluble subgroup of  $GL(n, \overline{\mathbb{F}})$  and let  $S$  be the soluble radical of  $\Delta$ . Then  $\Delta$  has a characteristic subgroup*



$\Delta_0 \geq S$  such that  $\Delta_0/S$  is a direct product of simple groups of Lie type of the same characteristic as  $\overline{\mathbb{F}}$  and  $|\Delta/\Delta_0| \leq f(n)$  (where  $f(n)$  is as in Proposition 88). Moreover the Lie rank of the simple factors appearing in  $\Delta_0/S$  is bounded by  $n$  and the number of simple factors is also at most  $n$ .

*Proof.* If  $\text{char}(\overline{\mathbb{F}}) = 0$  this is a theorem of Platonov (see [60]). Assume  $\text{char}(\overline{\mathbb{F}}) = p > 0$ . Let  $D$  be the Zariski closure of  $\Delta$ . Then  $D^0$  is soluble (see [60, Theorem 5.11]) and  $(D^0)\Delta = D$  hence  $\tilde{\Delta} = \Delta/(\Delta \cap D^0) \cong D/D^0$ . By a result of Platonov (see [60, Lemma 10.10]) we have  $D = (D^0)G$  where  $G$  is some finite subgroup of  $D$ , hence  $G/(G \cap D^0) \cong D/D^0$ . Now  $\tilde{\Delta}$  is isomorphic to a quotient of the finite group  $G \leq GL(n, \overline{\mathbb{F}})$  by a soluble normal subgroup. Therefore Proposition 88 implies that  $\tilde{\Delta}$  has a characteristic subgroup  $H$  of index at most  $f(n)$  such that  $H/\text{Sol}(\tilde{\Delta})$  is in  $\text{Lie}^*(p)$  (we can take  $H/\text{Sol}(\tilde{\Delta})$  to be the  $\text{Lie}^*(p)$  part of the socle of  $\tilde{\Delta}/\text{Sol}(\tilde{\Delta})$ ). Using [19, Theorem 3.4B] it follows that  $H/\text{Sol}(\tilde{\Delta})$  is isomorphic to a quotient of a finite subgroup of  $GL(n, \overline{\mathbb{F}}_p)$ . As in the proof of Proposition 92 we see that the number of simple factors in  $H/\text{Sol}(\tilde{\Delta})$  and their Lie ranks are bounded by  $n$ . Let  $\Delta_0$  be the subgroup of  $\Delta$  which corresponds to  $H$ . This is a characteristic subgroup since the kernel of the homomorphism  $\Delta \rightarrow (\tilde{\Delta}/\text{Sol}(\tilde{\Delta}))$  is  $\text{Sol}(\Delta)$ , which is characteristic in  $\Delta$ . We obtain our statement.  $\square$

Combining Corollary 111 and Lemma 112 we see that a non-growing subset  $\alpha \subset GL(n, \overline{\mathbb{F}})$  is covered by a few cosets of a soluble by  $\text{Lie}^*(p)$  normal subgroup of  $\langle \alpha \rangle$ . To obtain another such subgroup  $\Gamma$  for which  $\alpha^6 \text{Sol}(\Gamma)$  contains  $\Gamma$  we need a bit more work. The following two lemmas taken together describe the structure of a (possibly infinite) soluble by  $\text{Lie}^*(p)$  linear group.

**Lemma 113.** *Let  $S \leq GL(n, \overline{\mathbb{F}})$  be a soluble subgroup normalised by a subset  $\alpha \subseteq GL(n, \overline{\mathbb{F}})$ . Then there is a closed subgroup  $D \leq GL(n, \overline{\mathbb{F}})$  containing  $\alpha$  and  $S$ , and a homomorphism  $\phi : D \rightarrow GL(n', \overline{\mathbb{F}})$  such that  $\ker(\phi)$  is soluble, contains  $S$ , and  $n'$  depends only on  $n$ .*

*Proof.* If  $S$  is abelian then we consider the centralisers  $A = \mathcal{C}_{GL(n, \overline{\mathbb{F}})}(S)$  and  $B = \mathcal{C}_{GL(n, \overline{\mathbb{F}})}(A)$ . By [60, Theorem 6.2] we have homomorphisms

$$\phi_1 : \mathcal{N}_{GL(n, \overline{\mathbb{F}})}(A) \rightarrow GL(n^2, \overline{\mathbb{F}}), \quad \phi_2 : \mathcal{N}_{GL(n, \overline{\mathbb{F}})}(B) \rightarrow GL(n^2, \overline{\mathbb{F}})$$

whose kernels are precisely  $A$  and  $B$ . Note that  $A \cap B = \mathcal{Z}(A)$  contains  $S$ . Since  $\alpha$  normalises  $S$ , it also normalises  $A$  and  $B$ . The lemma holds

in this case with the following settings:

$$D = \mathcal{N}_{GL(n, \overline{\mathbb{F}})}(A) \cap \mathcal{N}_{GL(n, \overline{\mathbb{F}})}(B) ,$$

$$\phi = (\phi_1, \phi_2) : D \longrightarrow GL(n^2, \overline{\mathbb{F}}) \times GL(n^2, \overline{\mathbb{F}}) \leq GL(2n^2, \overline{\mathbb{F}}) .$$

In the general case we do induction on the derived length of  $S$ , which is bounded in terms of  $n$  [60, Theorem 3.7]. The commutator subgroup  $S^*$  is normalised by the subset  $\alpha^* = \alpha \cup S$ , we apply to them the induction hypothesis. We obtain a closed subgroup  $D^* \leq GL(n, \overline{\mathbb{F}})$  containing  $\alpha \cup S$  and a homomorphism  $\phi^* : D^* \rightarrow GL(m^*, \overline{\mathbb{F}})$  such that  $\ker(\phi^*)$  is soluble, contains  $S^*$ , and  $m^*$  depends only on  $n$ . The image  $\phi^*(S)$  is abelian and it is normalised by  $\phi^*(\alpha)$ . By the above settled case there is a closed subgroup  $D^{**} \leq GL(m^*, \overline{\mathbb{F}})$  containing  $\phi^*(\alpha)$  and a homomorphism  $\phi^{**} : D^{**} \rightarrow GL(m^{**}, \overline{\mathbb{F}})$  such that  $\ker(\phi^{**})$  is soluble, contains  $\phi^*(S)$ , and  $m^{**}$  depends only on  $m^*$ , hence only on  $n$ . We set

$$D = \phi^{*-1}(D^{**}) , \quad \phi = \phi^{**} \circ \phi^* , \quad m = m^{**} ,$$

the induction step is complete.  $\square$

**Lemma 114.** *Let  $\Lambda$  be a subgroup of  $GL(n, \overline{\mathbb{F}})$ ,  $\text{char}(\overline{\mathbb{F}}) = p$  and  $L$  a finite normal subgroup of  $\Lambda$  such that  $L$  is in  $\text{Lie}^*(p)$ . Then  $\Lambda/LC_\Lambda(L) \leq f(n^2)$  where  $f()$  is as in Proposition 88.*

*Proof.* By [60, Theorem 6.2]  $\Lambda/C_\Lambda(L)$  is a subgroup of  $GL(n^2, \overline{\mathbb{F}})$  hence by Proposition 88 it has a soluble by  $\text{Lie}^*(p)$  normal subgroup  $N$  of index at most  $f(n^2)$ . On the other hand  $\Lambda/C_\Lambda(L)$  is isomorphic to a subgroup  $A$  of  $\text{Aut}(L)$  containing  $\text{Inn}(L) \cong L$ . It is easy to see that the socle of  $A$  is  $\text{Inn}(L)$ . Therefore all soluble by  $\text{Lie}^*(p)$  normal subgroups of  $A$  are actually  $\text{Lie}^*(p)$  subgroups of  $\text{Inn}(L)$ . Our statement follows.  $\square$

We need two more auxiliary results on  $\text{Lie}^*(p)$  groups.

**Lemma 115.** *Let  $H$  be a normal subgroup of a group  $G$  and assume that  $H$  is a direct product of at most  $m$  finite simple groups of Lie type of rank at most  $m$ . Let  $\alpha$  be a symmetric subset of  $G$  covered by  $x$  cosets of  $H$ . If  $|\alpha| \geq |H|/y$  then  $H$  has a (possibly trivial) characteristic subgroup  $N$  such that  $N$  is contained in  $\alpha^6$  and  $|H/N| \leq (xy)^{Cm^2}$  for some constant  $C$ .*

*Proof.* If  $L$  is a simple direct factor of  $H$  and  $k = k(L)$  is the minimal degree of a non-trivial complex representation of  $L$  then by Proposition 76 we have  $|L| < k^{\frac{C}{3}m}$  for some absolute constant  $C$ . Let  $k_0 < k_1 < \dots$  be the different numbers  $k(L)$ . Define  $H_i$  as the product of the direct factors  $L$  for which  $k(L) \geq k_i$ . The  $H_i$  are

characteristic subgroups of  $H$ . By our assumptions for all indices  $i$  we have  $|\alpha^2 \cap H_i| \geq |\alpha|/x|H/H_i| \geq |H_i|/xy$ . By Proposition 75 if  $|\alpha^2 \cap H_i| > |H_i|/(k_i)^{1/3}$  then we have  $H_i \subseteq \alpha^6$ . Let  $j$  be the smallest index for which this holds. By the above for all  $i < j$  we have  $k_i \leq (xy)^3$  hence if  $L$  is a simple constituent of  $|H/H_j|$  then  $|L| < (xy)^{Cm}$ . Setting  $N = H_j$  we obtain that  $|H/N| \leq (xy)^{Cm^2}$ , as required.  $\square$

**Lemma 116.** *Let  $L = L_1 \times \cdots \times L_m$  be a direct product of simple groups of Lie type of rank at most  $r$ . Let  $\alpha$  be a symmetric generating set of  $L$  which projects onto all simple quotients of  $L$ . Then  $\alpha^{c(m,r)} = L$  where  $c(m,r)$  depends only on  $m$  and  $r$ .*

*Proof.* We need the following

**Claim 117.** *Let  $x = (x_1, \dots, x_t)$  be an element of a product  $L_1 \times \cdots \times L_t$  of simple groups of Lie type of rank at most  $r$  such that all  $x_i$  are non-trivial. Then each element of  $L_1 \times \cdots \times L_t$  is a product of at most  $Cr$  conjugates of  $x$  for an absolute constant  $C$ .*

For  $t = 1$  this is proved in [42] and the general case is an obvious consequence.

We prove the lemma by induction on  $m$ . It is clear that  $\alpha^2$  has two elements whose first projections are the same, hence  $\alpha^3$  contains a non-trivial element  $a = (a_1, \dots, a_m)$  such that  $a_1 = 1$ . Assume that  $a_{i+1}, \dots, a_m$  are the projections of  $a$  different from 1. By the induction hypothesis we know that  $\beta = \alpha^{c(m-1,r)}$  projects onto the quotient  $L/L_1$ . By the claim each element of  $L_{i+1} \times \cdots \times L_m$  is a product of at most  $Cr$  conjugates of  $a$  by elements of  $\beta$ , hence this subgroup is contained in  $(\alpha^3\beta^2)^{Cr}$ . Using again the induction hypothesis we see that  $\beta$  projects onto  $L_1 \times \cdots \times L_{m-1}$  hence  $L \leq \beta L_m \leq (\alpha^3\beta^3)^{Cr}$ . We obtain that  $L \leq \alpha^{3Cr(c(m-1,r)+1)}$  which completes the induction step.  $\square$

Finally we are ready to prove Theorem 10.

**Theorem 118.** *Let  $\alpha \subseteq GL(n, \overline{\mathbb{F}})$  be a finite symmetric subset such that  $|\alpha^3| \leq \mathcal{K}|\alpha|$  for some  $\mathcal{K} \geq \frac{3}{2}$ . Then there are normal subgroups  $S \leq \Gamma$  of  $\langle \alpha \rangle$  and a bound  $m$  depending only on  $n$  such that  $\Gamma \subseteq \alpha^6 S$ , the subset  $\alpha$  can be covered by  $\mathcal{K}^m$  cosets of  $\Gamma$ ,  $S$  is soluble, and the quotient group  $\Gamma/S$  is the product of finite simple groups of Lie type of the same characteristic as  $\overline{\mathbb{F}}$ . (In particular, in characteristic 0 we have  $\Gamma = S$ .) Moreover, the Lie rank of the simple factors appearing in  $\Gamma/S$  is bounded by  $n$ , and the number of factors is also at most  $n$ .*

*Proof.* If  $\text{char}(\overline{\mathbb{F}}) = 0$  then our statement follows from Corollary 111 and Lemma 112. Assume that  $\text{char}(\overline{\mathbb{F}}) = p > 0$ . Corollary 111 and

Lemma 112 imply that  $\Lambda = \langle \alpha \rangle$  has a normal subgroup  $\Delta$  such that  $\Delta/Sol(\Delta)$  is in  $Lie^*(p)$  and  $\alpha$  is covered by  $K^{a(n)}$  cosets of  $\Delta$  where  $a(n)$  depends on  $n$ . Moreover  $\Delta/Sol(\Delta)$  is the direct product  $L_1 \times \cdots \times L_t$  of at most  $n$  simple groups of Lie type of rank at most  $n$ . We set  $S = Sol(\Delta)$ . The proof of our theorem reduces to the following.

**Claim 119.** *The group  $\Lambda$  has a normal subgroup  $\Gamma$  such that  $\Delta \geq \Gamma \geq S$ ,  $S\alpha^6 \geq \Gamma$  and  $\alpha$  is covered by  $K^m$  cosets of  $\Gamma$ .*

To prove the claim, by Lemma 113 and Proposition 91 we might as well assume (at the cost of enlarging  $n$  and  $K$ ) that  $S = \{1\}$ , i.e.  $\Delta = L_1 \times \cdots \times L_t$ . In this case Proposition 114 implies that  $\Lambda$  has a normal subgroup  $H$  of index at most  $f(n^2)$  such that  $H$  is the direct product of  $\Delta$  and  $C = C_\Lambda(\Delta)$ . Set  $\gamma = \alpha^{2f(n^2)} \cap H$ . Slightly adjusting the proof of Proposition 94.(a) we see that  $\gamma$  generates  $H$  and  $|\gamma^3| \leq K_0|\gamma|$  where  $K_0 = K^{7f(n^2)}$ .

Denote by  $N_j$  the (unique) direct complement of  $L_j$  in  $H$ . Using Theorem 77 (as in the proof of Proposition 92) we see that for the quotients  $H/N_j \cong L_j$  we have two possibilities; either  $\gamma^3$  projects onto  $H/N_j$  (in which case  $|\gamma N_j/N_j| \geq |L_j|/K_0^2$  by Proposition 91) or  $|\gamma N_j/N_j| \leq K_0^{b(n)}$  where  $b(n)$  depends only on  $n$ . Let  $H/N_1, \dots, H/N_i$  be the quotients for which the first possibility holds and which also satisfy  $|L_j| > K_0^{b(n)+4}$ .

Since  $H/C$  is a direct product of nonabelian simple groups it follows that conjugation by  $\alpha$  permutes the simple factors, therefore it permutes the subgroups  $N_j$ . By an argument as in the proof of Proposition 94.(b) we see that the set  $\{N_1, \dots, N_i\}$  is invariant under conjugation by  $\alpha$ . Therefore  $N = N_1 \cap \cdots \cap N_i$  and  $I = N_{i+1} \cap \cdots \cap N_t$  are normal subgroups of  $\Lambda$ . By our assumptions  $\gamma^3$  projects onto all simple quotients of  $H/N$  and  $(\gamma N)/N$  generates this group. By Lemma 116 we see that  $\gamma^{c(n)}$  projects onto  $H/N$  where  $c(n)$  depends on  $n$ . This implies  $|\alpha|K^{d(n)} \geq |H/N|$  where  $d(n) = 2f(n^2)c(n)$ .

The subgroup  $D = I \cap \Delta = L_1 \times \cdots \times L_i$  is also normal in  $\Lambda$  and we have  $H/N \cong D$ , hence  $|\alpha|K^{d(n)} \geq |D|$ . By our assumptions  $\gamma$  projects onto at most  $K_0^{n(b(n)+4)} = K^{e(n)}$  elements of  $H/I$ . Since  $\alpha^2 \cap H \subseteq \gamma$ , the natural isomorphism between  $H/I$  and  $\Delta/D$  implies that  $\alpha^2 \cap \Delta$  projects onto at most  $K^{e(n)}$  elements of  $\Delta/D$ . Using Proposition 93 we see that  $\alpha$  is covered by  $K^{a(n)+e(n)}$  cosets of  $D$ . Since  $|\alpha| \geq |D|/K^{d(n)}$ , Lemma 115 implies that  $D$  has a characteristic subgroup  $\Gamma$  contained in  $\alpha^6$  such that  $|D/\Gamma| \leq K^{(a(n)+d(n)+e(n))Cn^2}$ . The subgroup  $\Gamma$  is normal in  $\Lambda$  and  $\alpha$  is covered by  $|D/\Gamma|K^{a(n)+e(n)}$  cosets of  $\Gamma$ . Our statement follows.  $\square$

Theorem 10 does not hold for all  $\mathcal{K} \geq 1$ . For example  $\alpha$  could be a subgroup of  $GL(n, \overline{\mathbb{F}})$  isomorphic to  $Alt(n)$ . However the structure of subsets  $\alpha$  with  $|\alpha^3| < \frac{3}{2}|\alpha|$  is completely described in Proposition 109.

#### 14. EXAMPLES

In this section we give some examples which show that the constant  $\varepsilon(r)$  for which Theorem 77 holds must be less than  $\frac{C}{r}$ . It will be convenient to rely on [3, Section 3] in describing our examples.

*Example 120.* Consider the group  $SL(n, q)$  where  $n \geq 3$  (which has Lie rank  $r = n - 1$ ). Let  $H$  be the subgroup of all diagonal matrices, this has order  $(q - 1)^{n-1}$ . If  $N$  denotes the subgroup of all monomial matrices then  $N/H \simeq S_n$ . Choose an element  $s$  of  $N$  projecting onto an  $n$ -cycle of  $N/H$ . If  $e_1, \dots, e_n$  is the standard basis of  $\mathbb{F}_q^n$ , consider the subgroup  $L_{1,2} \simeq SL(2, q)$  which fixes  $e_3, \dots, e_n$ . In [3, Theorem 3.1] a 3-element generating set  $\{a, b, c\}$  of  $L_{1,2}$  is chosen. As shown in [3]  $s, a, b$  and  $c$  generate  $SL(n, q)$  (moreover, the diameter of the corresponding Cayley graph is logarithmic).

Now  $s$  normalises the diagonal subgroup  $H$  and it is clear that  $a, b$  and  $c$  normalise a subgroup  $H_0$  of index  $(q - 1)^2$  in  $H$  (the group of diagonal matrices fixing  $e_1$  and  $e_2$ ). Our generating  $A$  set will consist of  $H, a, b, c$  and  $s$ . We claim that

$$|A^3| \leq |H|(3(q - 1)^2 + 58) + 64.$$

It is straightforward to see that

$$|A^3| \leq |H\{a, b, c, s\}H| + 57|H| + 64.$$

Since  $s$  normalises  $H$  we have  $|HsH| = |H|$ . Since  $a$  (resp.  $b$  and  $c$ ) normalises  $H_0$  we have  $|HaH| \leq |H|(q - 1)^2$  (and analogous inequalities hold for  $b$  and  $c$ ) which implies the claim.

Setting  $q = 3$  we obtain the generating set with  $|A^3| \leq 100|A|$  mentioned in the introduction.

Clearly, there are many ways in which the above construction can be extended. For example the full diagonal subgroup  $H$  can be replaced by its characteristic subgroups isomorphic to  $C_t^{n-1}$  where  $t$  divides  $q - 1$ . This way e.g. we can construct large families of generating sets of constant growth whenever  $q$  is odd.

It would be most interesting to find some essentially different families of examples of large generating sets of  $SL(n, q)$  with constant growth.

The above generating sets of “moderate growth” are “dense” subsets of the union of a few cosets of some subgroup. This can be avoided. Assume that  $q = 2^p$  where  $p \geq n$  is an odd prime. It is well-known that

all divisors of  $q - 1$  are greater than  $2p + 1$ . Replace  $H$  in the above construction by a subset  $P \subseteq H$  of the form  $\prod^{n-1} \{g, g^2, \dots, g^n\} \subseteq \prod^{n-1} C_{q-1} \simeq H$  which is invariant under conjugation by the cyclic element  $s$ . Now  $A = P \cup \{a, b, c, s\}$  is a generating set of size roughly  $n^{n-1}$  with  $A^3$  of size roughly  $n^n$ . It is easy to see that  $P$  is far from being a subgroup of  $SL(n, q)$ .

## 15. APPENDIX

In this appendix we prove rigorously the algebraic geometry facts used in the paper. For reference we use [27, Sections I.1, I.2, I.7 and II.3], and also [39, Section I.3]. Besides that, we need Proposition 121, which is a version of Bézout's theorem, stated and proved in [23].

Let  $\overline{\mathbb{F}}^m$  denote the  $m$ -dimensional affine space over the algebraically closed field  $\overline{\mathbb{F}}$ , and  $\mathbb{P}^m$  denote its projective closure. For a locally closed subset  $X \subseteq \overline{\mathbb{F}}^m$ , in this appendix  $\overline{X}$  denotes (as before) the closure of  $X$  in  $\overline{\mathbb{F}}^m$ , and  $\overline{X}^{\mathbb{P}^m}$  denotes the closure of  $X$  in  $\mathbb{P}^m$ . Similarly,  $\deg(X)$  and  $\deg(\overline{X})$  denotes the degrees in the sense of Definition 15, and  $\deg_{\mathbb{P}^m}(\overline{X}^{\mathbb{P}^m})$  denotes the degree of the projective variety  $\overline{X}^{\mathbb{P}^m} \subseteq \mathbb{P}^m$  in the sense of [27, Section I.7]. Note, that both notions of degree depend not only on the isomorphism type of  $X$ , but also on the particular embedding of  $X$  into the affine (or projective) space.

**Proposition 121** (Fulton, see [23]). *Let  $P, Q$  be irreducible closed subsets of the projective space  $\mathbb{P}^m$ , and let  $Z_1, \dots, Z_k$  be the irreducible components of  $P \cap Q$ . Then*

$$\deg_{\mathbb{P}^m}(P) \cdot \deg_{\mathbb{P}^m}(Q) \geq \sum_{i=1}^k \deg_{\mathbb{P}^m}(Z_i) .$$

Definitions 11, 12, 13 and 14 are standard, we do not comment on them. On the other hand, the degree is usually defined for projective varieties, and in Definition 15 we deal with locally closed subsets of  $\overline{\mathbb{F}}^m$ . The connection with the usual notions is explained by the following:

**Proposition 122.** *For a locally closed subset  $X \subseteq \overline{\mathbb{F}}^m$  we have*

$$\dim(X) = \dim(\overline{X}) = \dim(\overline{X}^{\mathbb{P}^m}) ,$$

$$\deg(X) = \deg(\overline{X}) = \deg_{\mathbb{P}^m}(\overline{X}^{\mathbb{P}^m}) .$$

*Moreover,  $X$  is irreducible iff  $\overline{X}^{\mathbb{P}^m}$  is irreducible.*

*Proof.* The last statement follows from [27, Ex.I.1.6]. Then it is enough to prove the two equalities for irreducible  $X$ . So we assume that  $X$  is

irreducible. The equality of dimensions is [27, Ex.I.2.7]. Let  $\mathcal{L}$  denote the collection of affine subspaces  $L \subseteq \overline{\mathbb{F}}^m$  of dimension  $m - \dim(X)$ . For all members  $L \in \mathcal{L}$ , the intersection  $\overline{L}^{\mathbb{P}^m} \cap \overline{X}^{\mathbb{P}^m}$  is either infinite, or it has at most  $\deg_{\mathbb{P}^m}(\overline{X}^{\mathbb{P}^m})$  points. Moreover, for almost all  $L$  the intersection  $\overline{L}^{\mathbb{P}^m} \cap \overline{X}^{\mathbb{P}^m}$  have exactly  $\deg_{\mathbb{P}^m}(\overline{X}^{\mathbb{P}^m})$  points and  $\overline{L}^{\mathbb{P}^m}$  avoids the smaller dimensional boundary  $\overline{X}^{\mathbb{P}^m} \setminus X$ . This proves that  $\deg(X) = \deg_{\mathbb{P}^m}(\overline{X}^{\mathbb{P}^m})$ . The same argument applied to  $\overline{X}$  completes the proof.  $\square$

Remark 16 follows immediately from our definition of  $\deg(X)$ , as a single point has degree 1. Definition 17 and Remark 18 are standard, we do not comment on them.

*Proof of Fact 19.* (a), (b) and (c) follows from Proposition 122 and the analogous statements for projective varieties. (e) follows from [27, Ex.I.1.6] and from the definition of the dimension.

Combining Proposition 122 with  $\overline{X \cup Y}^{\mathbb{P}^m} = \overline{X}^{\mathbb{P}^m} \cup \overline{Y}^{\mathbb{P}^m}$ ,  $\overline{X \cap Y}^{\mathbb{P}^m} \subseteq \overline{X}^{\mathbb{P}^m} \cap \overline{Y}^{\mathbb{P}^m}$ ,  $X \setminus \overline{Y} \subseteq \overline{X}^{\mathbb{P}^m} \setminus \overline{Y}^{\mathbb{P}^m}$  we obtain most of (d), with the exception of its last equality. Next we consider the intersection

$$(X \times \overline{\mathbb{F}}^m) \cap (\overline{\mathbb{F}}^m \times Y) = X \times Y \subseteq \overline{\mathbb{F}}^{2m}.$$

Taking closures in  $\mathbb{P}^{2m}$  and applying [27, Theorem I.7.7] we obtain the last equality of (d).

If  $X$  and  $Y$  are irreducible then  $\overline{X} \times \overline{Y} = \overline{X \times Y}$  is irreducible by [27, Ex.I.3.15(d)], hence (f) follows from [27, Ex.I.1.6].

Next we introduce two invariants of closed subsets. If  $Z \subseteq \overline{\mathbb{F}}^m$  is a closed set with irreducible decomposition  $Z = \bigcup_i Z_i$  then we define

$$N(Z) = \sum_i (d+1)^{\dim(Z_i)} \deg(Z_i) \quad \text{and} \quad D(Z) = \sum_i d^{\dim(Z_i)} \deg(Z_i).$$

Let  $F$  be the zero set of a polynomial of degree  $d$  which does not vanish identically on  $Z$ . By Proposition 121 we have  $N(Z_i \cap F) < N(Z_i)$  and  $D(Z_i \cap F) \leq D(Z_i)$  whenever  $Z_i \not\subseteq F$ , therefore  $N(Z \cap F) < N(Z)$  and  $D(Z \cap F) \leq D(Z)$ . To obtain  $X$  we start from  $\overline{\mathbb{F}}^m$ , and add the equations of  $X$  of degree  $d$  one by one, until their common zero locus becomes  $X$ . We obtain that  $\deg(X) \leq D(X) \leq D(\overline{\mathbb{F}}^m) = d^m$ , and the invariant  $N$  decreases in each step, i.e. we need at most  $N(\overline{\mathbb{F}}^m) = (d+1)^m$  equations. One direction of (g) is proved. The other direction of (g) follows from [39, Section I.3] (the construction of the Chow variety).  $\square$

*Proof of Fact 20.* Let  $X \subseteq \overline{\mathbb{F}}^n$  and  $Y \subseteq \overline{\mathbb{F}}^m$  denote the ambient spaces (see the note after Definition 11), and let  $\pi : \overline{\mathbb{F}}^n \times \overline{\mathbb{F}}^m \rightarrow \overline{\mathbb{F}}^m$  denote the projection to the second factor. Note that  $\Gamma_f$  is isomorphic to  $X$  (via the first projection), and  $f(X) = \pi(\Gamma_f)$ .

We already proved (a) with the exception of the degree estimates which we postpone for a while.

In the proof of (b) we may (and do) assume that  $X$  is irreducible. If  $\overline{f(X)} = A \cup B$  were a proper decomposition into closed subsets then  $X = f^{-1}(A) \cup f^{-1}(B)$  would also be a proper decomposition, a contradiction. Hence  $\overline{f(X)}$  is also irreducible. By [27, Ex.II.3.19(b)] the subset  $f(X)$  contains a dense open subset  $U \subseteq \overline{f(X)}$ . It remains to estimate  $\deg(f)$ . Let  $L \subseteq \overline{\mathbb{F}}^m$  be an affine subspace of dimension  $m - \dim(\overline{f(X)})$  which intersects  $U$  in exactly  $\deg(U) = \deg(\overline{f(X)})$  points (see Definition 15 and Fact 19.(a)). Then  $\pi^{-1}(L)$  is an affine subspace, hence  $\deg(f) = \deg(\Gamma_f) \geq \deg(\Gamma_f \cap \pi^{-1}(L))$ . But  $\Gamma_f \cap \pi^{-1}(L)$  is isomorphic to  $f^{-1}(L) = f^{-1}(U \cap L)$ , hence it has at least  $\deg(\overline{f(X)})$  connected components. This implies that  $\deg(f) \geq \deg(\overline{f(X)})$ , (b) is proved.

Next we prove (c). We know that  $f^{-1}(T)$  is isomorphic to  $\Gamma_f \cap \pi^{-1}(T)$ , and  $\pi^{-1}(T) = \overline{\mathbb{F}}^n \times T$  have degree  $\deg(T)$  by Fact 19.(d). Then  $\deg(f^{-1}(T)) \leq \deg(T) \deg(f)$  by Fact 19.(d). In the special case  $T = \{y\}$  we obtain  $\deg(f^{-1}(y)) \leq \deg(f)$ , which completes the proof of (c).

The closed complement considered in (d) is the union of a number of the locally closed subsets of (a), hence its degree bound follows immediately from (a). So (d) is proved.

[27, Ex.II.3.22(b)] contains the inequality of (e) as well as the openness and denseness of  $X_{\min}$ . The difference  $X \setminus X_{\min}$  is the inverse image of the union of a number of the locally closed subsets of (a), hence its degree bound follows from (a) and (c). This proves (e).

In (f), the graph of the restricted morphism  $f|_S$  is  $\Gamma_f \cap (S \times \overline{\mathbb{F}}^m)$ . By Fact 19.(d) it has degree at most  $\deg(\Gamma_f) \deg(S \times \overline{\mathbb{F}}^m) = \deg(f) \deg(S)$ . Moreover, if  $S$  is an irreducible component of  $X$  then the graph of  $f|_S$  is the corresponding component of  $\Gamma_f$ . This proves (f).  $\square$

*Proof of Fact 20.(a), counting the sheep.* First we bound the number of the parts in the partitions of  $Z$ . In the proof we partition  $Z'_j$  in at most  $d+1$  steps. In the very first step we subdivide  $Z'_j$  into  $(d+1)^2 + 2$  parts, and the algorithm stops in two of them. Suppose that  $C$  is a partition class constructed before the  $(l-1)$ -th polynomial division and



the algorithm did not stop in  $C$ . Before the  $l$ -th division we subdivide  $C$  into  $d+2$  parts, in one of them the algorithm stops, in the other  $d+1$  it continues. Altogether we cut  $Z'_j$  into at most  $2 + \sum_{l=1}^d (d+1)^{l+1} \leq (d+2)^{d+1}$  pieces, and we repeat this cutting less than  $(d+1)^{k+1}$  times. Hence we obtain altogether at most  $(d+2)^{(d+1)((d+1)^{k+1}-1)}$  parts  $Z_i$ . Finally we cut each  $Z_i$  again into at most  $d+2$  parts.

Let  $p(t, \underline{x})$  and  $q(t, \underline{x})$  be polynomials of  $t$ -degree at most  $d$  and  $\underline{x}$ -degree at most  $e$ . We divide by the leading  $t$ -coefficients, then all  $t$ -coefficients are rational functions of degree at most (with nonstandard notation)  $e/e$ . We do polynomial division: both the quotient and the remainder have coefficients of degree at most  $e^2/e^2$ . We run Euclid's algorithm for  $p$  and  $q$ . We do at most  $d$  divisions. In each quotient and in each remainder the  $t$ -coefficients have degrees at most  $e^{2^d}/e^{2^d}$ . Then we multiply through with the denominators.

In the proof of Claim 21.(a) we run Euclid's algorithm at most  $(d+1)^{k+1} - 1$  times. So each polynomial we encounter (including the  $P_i$ ) has  $t$ -degree at most  $d$  and  $\underline{x}$ -degree at most  $d^{((d+1)^{k+1}-1)2^d}$ , hence their total degree is at most  $d^{(d+1)^{k+1}2^d}$ . In the proof of Claim 21.(b) each  $Z_i$  is subdivided into at most  $d+2$  locally closed subsets defined via the vanishing or non-vanishing of several  $k$ -variate polynomials of degree at most  $d^{(d+1)^{k+1}2^d}$ .

In the proof of Fact 20.(a) we start from  $\overline{f(X)}$  (which has degree at most  $\deg(f)$ ), and apply Claim 21 at most  $\dim(X) + \deg(f) - 1$  times. Each time we subdivide each locally closed subset into at most  $\Phi(\Phi(\dots \Phi(\deg(f)) \dots))$  pieces and each piece is defined with equations of degree at most  $\Phi(\Phi(\dots \Phi(\deg(f)) \dots))$ . At the end we obtain altogether at most  $D$  locally closed parts and their degrees are at most  $D$  (see Fact 19.(g)).

Finally, in Fact 20.(d) the subset in question is the union of a number of the locally closed subsets of (a), and the subset in Fact 20.(e) is the inverse image of such a union. Hence their degrees are at most  $D^2$  and  $D^2 \deg(f)$  respectively.  $\square$

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